Complete surfaces with negative extrinsic curvature

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Abstract

N. V. Efimov [Efi64] proved that there is no complete, smooth surface in $\mathbf{R}^3$ with uniformly negative curvature. We extend this to isometric immersions in a 3-manifold with pinched curvature: if M^3 has sectional curvature between two constants K_2 and K_3, then there exists $K_1 < \min(K_2, 0)$ such that M contains no smooth, complete immersed surface with curvature below K_1. Optimal values of K_1 are determined. This results rests on a phenomenon of propagations for degenerations of solutions of hyperbolic Monge-Ampère equations.

Résumé

N. V. Efimov [Efi64] a montré qu'il n'existe pas de surface complète à courbure uniformément négative dans $\mathbf{R}^3$. On étend ce résultat aux immersions isométriques dans les 3-variétés à courbure pincée: si M^3 a sa courbure sectionnelle comprise entre deux constantes K_2 et K_3, alors il existe une constante $K_1 < \min(K_2, 0)$ telle que M ne contient pas de surface immergée complète et régulière à courbure inférieure à K_1. Des valeurs optimales de K_1 sont déterminées. Ce résultat repose sur un phénomène de propagation pour les dégénérescences de solutions d'équations de Monge-Ampère hyperboliques.

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Hilbert [Hil01] proved that there is no smooth isometric immersion of the hyperbolic plane H^2 into the Euclidean 3-space $\mathbf{R}^3$. This was extended by Efimov, who replaced H^2 by any complete surface with uniformly negative curvature:

Theorem 0.1 (N. V. Efimov [Efi64]). *Let (Σ, σ) be a smooth, complete Riemannian surface with curvature $K \le -1$. Then (Σ, σ) has no C^2 isometric immersion into $\mathbf{R}^3$.*

This result was proved using some subtle geometric constructions, strongly based on the Euclidean structure of the target space. More details can be found in [Efi68a], [Klo72] or in [BS92, Roz92], and some extensions and related results in [Efi68b, Efi62, Efi66].

It seems rather natural to try to extend Hilbert's result further by replacing also $\mathbf{R}^3$ by a Riemannian manifold. This was started in [Sch99], where the target space can be a Riemannian or Lorentzian 3-dimensional space-form. The present paper treats the case where it is a Riemannian manifold with pinched curvature.

Theorem 0.2. *Let (M, μ) be a complete Riemannian 3-manifold, with sectional curvature K_M between two constants $K_2 \le K_3$. Let (Σ, σ) be a complete Riemannian surface, with curvature $K_\Sigma \le K_1$, with $K_1 < 0$, $K_1 < K_2 \le K_3$, and:*

- *either $K_3 \ge 0$ and*
$$(K_3 - K_2)^2 < 16|K_1|(K_2 - K_1) \ ;$$

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- *or $K_3 \leq 0$ and*

$$(K_3 - K_2)^2 < 16(K_3 - K_1)(K_2 - K_1) \ .$$

Finally, suppose that $\|\nabla(K_\Sigma^{-1/2})\|$ and $\|\nabla K_M\|$ are bounded. Then there exists no C^3 isometric immersion from (Σ, σ) into (M, μ).

The meaning of "$\|\nabla K_M\|$ bounded" demands some precisions. Let $m \in M$, let P be a 2-plane in $T_m M$, and let $c : [0, 1] \to M$ be a smooth curve with $c(0) = m$. For $t \in [0, 1]$, call P_t the parallel transport of P at $c(t)$ along $c([0, t])$, and let $K(t)$ be the sectional curvature of M on P_t. Then our hypothesis is that $|K'(t)|$ is bounded by some fixed constant.

The proof of theorem 0.2 rests on two ideas, one of a geometric and the other of an analytical nature.

The geometric point concerns which objects, induced on a surface by an immersion, are to be considered. Of course, one could consider the induced metric – also called the first fundamental form I of the immersion – along with its Levi-Civita connection ∇ and the "Weingarten operator" B, which satisfies what can be described as a Monge-Ampère equation of hyperbolic type: $\det(B)$ is equal to the extrinsic curvature of the immersion (which is negative here), while $d^\nabla B$ is equal to another term given by the Coddazi equation, which is bounded. There are some "dual" objects, however, which are of greater use: the third fundamental form $I\!I\!I$ of the surface, and the inverse $\tilde{B}$ of B. The "new" point is that the "right" connection to use is not the Levi-Civita connection of $I\!I\!I$, but rather another connection, called $\tilde{\nabla}$, which is compatible with $I\!I\!I$ and has bounded torsion. $\tilde{B}$ then satisfies a very simple equation: $\det(\tilde{B})$ is again given by the extrinsic curvature, while $d^{\tilde{\nabla}} \tilde{B} = 0$. When the ambiant space has constant curvature, $\tilde{\nabla}$ is indeed the Levi-Civita connection of $I\!I\!I$.

The analytical fact which is important in the proof is about propagations of degenerations of sequences of solutions of some hyperbolic Monge-Ampère equations. Remember again that isometric immersions of surfaces are described analytically as solutions of Monge-Ampère equations. When the extrinsic curvature of the immersed surface is positive, the equations are elliptic, and this case is rather well understood [Pog73, CNS84a, CNS87, CNS85, CNS84b, Lab89, Sch96, LS99]. A fundamental point is that solutions of those equations have no isolated singularities: rather, if a sequence of solution has a limit which is degenerate at a point, then (for some subsequence) the same happens along a geodesic. This phenomenon has been studied completely by F. Labourie in [Lab87, Lab89, Lab97] (see [BK96] for some related problems). It is interesting to remark that, for complex Monge-Ampère solutions, the geometric nature of the locus of degeneration of sequences of solutions also plays a major role (see e.g. [Nad90]).

On the other hand, it has been knows since [Roz62] that surfaces with negative curvature in $\mathbf{R}^3$ can have an isolated singularity. Nonetheless, a phenomenon of propagation of degenerations of sequences of solutions of hyperbolic Monge-Ampère equations appears when the singularities are supposed to be bad enough. Here is an example of such a result.

Theorem 0.3 ([Sch99]). *Let D be a disk with a smooth Riemannian metric g with curvature $K < -1$, and let $(\phi_n)_{n \in \mathbf{N}}$ be a sequence of isometric immersions of (D, g) into $\mathbf{R}^3$. Let $x_0 \in D$, and let $y_0 \in \mathbf{R}^3$ be such that, for all n, $\phi_n(x_0) = y_0$. Suppose that (ϕ_n) is degenerate at x_0, in the sense that there exists a geodesic segment γ_0 with $\gamma_0(0) = x_0$ such that:*

$$\forall \epsilon > 0, \exists n \in \mathbf{N}, \int_0^\epsilon (I\!I\!I_n(\gamma_0'(s), \gamma_0'(s))^{1/2} ds \geq \frac{1}{\epsilon} \ .$$

Then there exists a subsequence $(\psi_n)_{n \in \mathbf{N}}$ of $(\phi_n)_{n \in \mathbf{N}}$ and a maximal geodesic segment g going through x_0 such that (ψ_n) is degenerate along g :

$$\forall \epsilon > 0, \exists N \in \mathbf{N}, \forall n \geq N, \forall x \in g, \exists y \in B_\mu(x, \epsilon), H_n(y) \geq 1/\epsilon \ .$$

Moreover $(\psi_{n|g})_{n \in \mathbf{N}}$ converges C^0 towards an isometry from g to a geodesic segment of $\mathbf{R}^3$.

This kind of propagation is essentially responsible for a crucial point of the proof of theorem 0.2, namely that $(\Sigma, I\!I\!I, \tilde{\nabla})$ is "convex" in a precise sense (see the next section). This fact, however, is somewhat hidden in the present proof, because a "shortcut" is used to obtain more rapidly this convexity

result. The reader is refered to [Sch99], where a special case (when the ambiant space has constant curvature) is proved using an analog of theorem 0.3. The resulting arguments are longer and more technical, but perhaps more illuminating than those given here.

It is not clear whether the hypothesis concerning the gradients of the curvature are really necessary here. On the other hand, the inequlities on K_1, K_2 and K_3 are more or less optimal, as is pointed out in section 8 using some examples.

Note that a nice analog of theorem 0.1 has been given by Smyth and Xavier [SX87] in higher dimension, for hypersurfaces with Ricci curvature conditions in $\mathbf{R}^{n+1}$ ($n \geq 3$). Some related results have also been given by Smyth [Smy92] in S^{n+1}. The approach they use, however, is very different from the path followed here – and it does not seem to work at all for surfaces. It would be most interesting to know whether something like the results of [SX87] applies to hypersurfaces in Riemannian manifolds.

1 How the proof works

The proof of theorem 0.2 happens almost entirely on Σ with its third fundamental form, along with a compatible connection $\tilde{\nabla}$ which is defined in section 2. $\tilde{\nabla}$ is the Levi-Civita connection of $I\!I\!I$ when M has constant curvature, but in general it has non-zero torsion. Its torsion, however, is bounded. Section 2 contains the proof of the following lemma, describing the basic geometric properties of $\tilde{\nabla}$.

Lemma 1.1. *Under the hypothesis of theorem 0.2, $\tilde{\nabla}$ is compatible with $I\!I\!I$, and has torsion τ bounded above by a constant τ_0. Its curvature $\tilde{K}$ is bounded between two positive constant:*

$$K_5 \geq \tilde{K} \geq K_4 > 0 \ .$$

Moreover:

$$4K_4 > \tau_0^2 \ .$$

We will also use the asymptotic directions of the immersion. More precisely, we can suppose that Σ is simply connected (otherwise consider its universal cover, which again has an isometric immersion into M). Therefore, we can choose two vector fields U and V, parallel to the asymptotic directions of the immersion, with unit norm for $I\!I\!I$. Since U and V are never parallel, we also demand that $\angle(U, V) \in (0, \pi)$. Section 2 repeats this definition, and contains the proof of the next lemma, about some key properties of U and V.

Lemma 1.2. *There exists a constant $\tau_1 > 0$ such that the asymptotic vectors U and V satisfy:*

$$\|\tilde{\nabla}_U V\| \leq \tau_1 \sin(\angle(U, V)) \ , \ \ \|\tilde{\nabla}_V U\| \leq \tau_1 \sin(\angle(U, V)) \ .$$

Section 3 contains some technical propositions concerning surfaces with connections having bounded torsion. Section 4 is about an amusing technical lemma which states that, if an asymptotic curve is "almost closed", then a propagation phenomenon happens. This is used in sections 5 and 6, which contain what is maybe the central point of this paper. One must first define the convexity of a (non-complete) surface in the following fairly natural way, basically stating that a geodesic segment can not touch the boundary at an interior point:

Definition 1.3. *Let $(S, \partial S)$ be a surface, with a metric g and a compatible connection D. We say that S is* **convex** *if, when $(\gamma_n)_{n\in\mathbf{N}}$ is a sequence of geodesic segments, $\gamma_n : [0, L] \to \Sigma$, such that $(\gamma_n(t))$ converges in $\overline{\Sigma}$ for each $t \in [0, L]$, and when there exists $t_0 \in]0, L[$ such that $\lim_{n\to\infty} \gamma_n(t_0) \in \partial\Sigma$, then $\lim_{n\to\infty} \gamma_n(t) \in \partial\Sigma$ for all $t \in [0, L]$.*

Then:

Lemma 1.4. *Σ, with $I\!I\!I$ and $\tilde{\nabla}$, is convex.*

To reach this goal, we define a specific notion of "concavity" of $\partial_{I\!I\!I}\Sigma$, and then prove that "concave" points are not possible. The convexity of Σ will then follow. First we choose positive real numbers k and C and a point $x \in \partial_{I\!I\!I}\Sigma$.

Definition 1.5. *A* (k, C)**-concave map** *at* x *is a map* $\phi : [-d, d] \times [0, d] \to \overline{\Sigma}$*, with* $d > 0$*, such that:*

- $\phi(0,0) = x$*, and* $\phi([-d,d]\times[0,d]\setminus(0,0)) \subset \Sigma$*,* ϕ *being a smooth diffeomorphism on its image outside* $(0,0)$*;*
- *for each* $y \in (0, d]$*, the curve* $\phi([-d, d] \times \{y\})$ *has geodesic curvature* κ *between* k *and* Ck*, with its convex side towards* x*, and* $|\partial_2\kappa| \leq C$*;*
- *at each point of* $[-d, d] \times [0, d] \setminus (0, 0)$*,* $\partial_1\phi$ *is orthogonal to* $\partial_2\phi$*, and* $1 \leq \|\partial_1\phi\|, \|\partial_2\phi\| \leq C$*.*

d *is called the* **diameter** *of* ϕ *and is written as* $\mathrm{diam}(\phi)$*.*

Definition 1.6. *Let* $x_0 \in \partial_{I\!I\!I}\Sigma$*.* Σ *is* (k, C)**-concave** *at* x_0 *if there exists a* (k, C)*-concave map* ϕ *at* x_0*.* Σ *is* **concave** *at* x_0 *if it is* (k, C)*-concave for some* $k > 0$ *and* $C > 0$*.*

The point of this definition is the following result, which is proved in section 5 using a technical lemma from section 4:

Lemma 1.7. *Under the hypothesis of theorem 0.2,* $\partial_{I\!I\!I}\Sigma$ *has no concave point.*

On the other hand, it is proved in section 6 that:

Lemma 1.8. *If* Σ *has no concave point, then it is convex.*

The proof of lemma 1.4 clearly follows from those two lemmas. It is then proved in section 7 that:

Lemma 1.9. *Under the hypothesis of theorem 0.2, if* $(\Sigma, I\!I\!I, \tilde{\nabla})$ *is convex, then it has bounded area.*

A contradiction will follow, because, by the Gauss formula, the ratio of the area elements on Σ for I and for $I\!I\!I$ is equal to the absolute value of the extrinsic curvature of the immersion, which is supposed to be bounded away from 0 in theorem 0.2; and the area of (Σ, I) is infinite because (Σ, I) is complete, simply connected, and with negative curvature.

Conventions: in the whole paper, if $c : [a, b] \to \Sigma$ is a piecewise smooth curve, and if $W \in T_{c(a)}\Sigma$, we let $\Pi(c; W)$ be the parallel transport of W at $c(b)$ along c. Unless otherwise stated, all curves are parametrized at unit speed.

2 Isometric immersions of surfaces

This section contains some elementary results concerning the objects induced on a Σ by an immersion in a Riemannian 3-space M. We call I the induced metric, ∇ its Levi-Civita connection, and ∇^M that of M.

We suppose that Σ is contractible and oriented – otherwise, consider its universal cover. We can therefore choose a unit normal vector field N to Σ, and define a bundle morphism (the "shape operator"):

$$\begin{array}{rrl} B: & T\Sigma \to & T\Sigma \\ & x \mapsto & \nabla^M_x N \ . \end{array}$$

It easy to check that B is symmetric. From there follows the definition of the third fundamental form of the immersion:

$$\forall s \in \Sigma,\ \forall x, y \in T_s\Sigma,\ I\!I\!I(x, y) = I(Bx, By) \ .$$

If $M = \mathbf{R}^3$, then $I\!I\!I$ is the pull-back of the canonical metric on S^2 by the Gauss map.

Let R be the Riemann curvature tensor of M. Then B satisfies the following classical equations (see [GHL87] or [Spi75], vol. III):

$$\forall s \in \Sigma,\ \forall x, y \in T_s\Sigma,\ (d^\nabla B)(x, y) = -R_{x,y}n \ ,$$

which is known as the Codazzi-Mainardi equation, and the Gauss equation:

$$\forall s \in \Sigma,\ \det(B_s) = K_e := K(s) - K_M(T_s\Sigma)\ ,$$

where $K(s)$ is the curvature of ∇ at s.

The main point of this section is that the immersion also defines on Σ a connection which is compatible with $I\!I\!I$, but in general has torsion.

Definition 2.1. *Let $\tilde\nabla$ be the connection defined on Σ by:*

$$\tilde\nabla_x y = B^{-1}\nabla_x(By)\ .$$

Remember that the torsion of a connection is a 2-form with value in the tangent space, which is defined as:

$$\tau(x,y) := \tilde\nabla_x y - \tilde\nabla_y x - [x,y]$$

The Levi-Civita connection of a metric is defined as the only compatible connection with zero torsion. Note that, for Riemannian surfaces, 2-forms can be identified with functions, so we will often here consider the torsion τ as a vector field on Σ. This identification will always be made using, as a Riemannian metric, the third fundamental form $I\!I\!I$.

The main property of $\tilde\nabla$ is given in the following proposition.

Proposition 2.2. *$\tilde\nabla$ is compatible with $I\!I\!I$. Its torsion is bounded by:*

$$\forall s \in \Sigma,\ \forall x,y \in T_s\Sigma,\ \|\tau(x,y)\|_{I\!I\!I} \leq \|(d^\nabla B)(x,y)\|_I\ .$$

Proof. Let x, y and z be three vector fields on Σ. Then:

$$\begin{aligned} x.I\!I\!I(y,z) &= x.I(By,Bz) \\ &= I(\nabla_x(By),Bz) + I(By,\nabla_z(Bz)) \\ &= I\!I\!I(B^{-1}\nabla_x(By),z) + I\!I\!I(y,B^{-1}\nabla_x(Bz)) \\ &= I\!I\!I(\tilde\nabla_x y,z) + I\!I\!I(y,\tilde\nabla_x z)\ , \end{aligned}$$

so that $\tilde\nabla$ is compatible with $I\!I\!I$.

From the definition of $\tilde\nabla$:

$$\begin{aligned} \tau(X,Y) &= \tilde\nabla_X Y - \tilde\nabla_Y X - [X,Y] \\ &= B^{-1}\nabla_X(BY) - B^{-1}\nabla_Y(BX) - [X,Y] \\ &= B^{-1}(d^\nabla B)(X,Y)\ , \end{aligned}$$

and this shows that:

$$\begin{aligned} \|\tau\|^2_{I\!I\!I} &= I\!I\!I(B^{-1}(d^\nabla B)(X,Y), B^{-1}(d^\nabla B)(X,Y)) \\ &= I((d^\nabla B)(X,Y),(d^\nabla B)(X,Y)) \\ &= \|(d^\nabla B)(X,Y)\|^2_I\ , \end{aligned}$$

and the result follows. □

As a consequence, $\tilde\nabla$ is the Levi-Civita connection of $I\!I\!I$ when M has constant curvature. When M does not have constant curvature, the previous proposition leads to the following control on the torsion of $\tilde\nabla$:

Proposition 2.3. *If (Σ, I) has curvature $K \leq K_1$, then, at any point $s \in \Sigma$, the torsion of $\tilde\nabla$ is bounded by:*

$$\|\tau\|_{I\!I\!I} \leq \tau_0(\phi(s)) := \frac{K_M - K_m}{2\sqrt{(K_m-K_1)(K_M-K_1)}}\ ,$$

where K_m and K_M are the minimum and the maximum of the sectional curvatures of M on tangent 2-planes at $\phi(s)$.

Proof. Let (e_1, e_2) be the orthonormal basis of $T_s\Sigma$ for I which diagonalizes B, and let k_1, k_2 be the associated eigenvalues. We need to prove the upper bound above with τ replaced by $\tau(e_1, e_2)/(k_1 k_2)$, because τ is skew-adjoint, and $((1/k_1)e_1, (1/k_2)e_2)$ is an orthonormal basis of $T_s\Sigma$ for $I\!I\!I$.

According to the previous proposition and to (1), it is enough to prove that, under our curvature assumptions, for any $m \in M$ and for any orthonormal basis (x, y, n) of $T_m M$:

$$\frac{\|R(x,y)n\|_I}{K(x,y) - K_1} \leq \tau_0 \ ,$$

where $K(x, y)$ is the sectional curvature of M on the 2-plane generated by x and y. Let $\check{R} : \Lambda^2 M \to \Lambda^2 M$ the curvature operator, and μ the metric on $\Lambda^2 M$ coming from the metric on M. We need to prove that, for any $m \in M$, if, when $v, w \in \Lambda^2_m M$ are orthogonal and have unit norm, $K_m \leq \mu(\check{R}v, v) \leq K_M$, then, with the same hypothesis on v and w, we have:

$$\left| \frac{\mu(\check{R}v, w)}{\mu(\check{R}v, v) - K_1} \right| \leq \tau_0(m) \ .$$

Let $m \in M$, and let $P \subset \Lambda^2_m M$ be a 2-plane. Denote by Q the restriction of $\check{R}$ to P followed by the orthonormal projection on P, p_1, p_2 its eigenvectors, and q_1, q_2 its eigenvalues. If $v, w \in P$ are orthogonal with unit norm, they can be written as $v = \cos(\theta)p_1 + \sin(\theta)p_2$ and $w = \sin(\theta)p_1 - \cos(\theta)p_2$, so that:

$$\frac{\mu(Qv, w)}{\mu(Qv, v) - K_1} = \frac{(q_1 - q_2)\cos(\theta)\sin(\theta)}{q_1 \cos^2(\theta) + q_2 \sin^2(\theta) - K_1} \ .$$

If now $\alpha := \cos^2(\theta)$, we find that:

$$\left| \frac{\mu(Qv, w)}{\mu(Qv, v) - K_1} \right|^2 = \frac{(q_1 - q_2)^2 \alpha (1 - \alpha)}{((q_1 - q_2)\alpha + q_2 - K_1)^2} \ .$$

This is maximal when:

$$\alpha = \frac{K_1 - q_2}{2K_1 - q_1 - q_2}$$

(which is in $[0, 1]$ and corresponds to a possible value of $\cos(\theta)$). Replacing α by this value in (1) shows that:

$$\left| \frac{\mu(Qv, w)}{\mu(Qv, v) - K_1} \right| \leq \frac{|q_1 - q_2|}{2\sqrt{(q_1 - K_1)(q_2 - K_1)}} \ .$$

Since the right side is maximal for $\{q_1, q_2\} = \{K_m, K_M\}$, we find the upper bound we need for

$$\frac{\mu(Qv, w)}{\mu(Qv, v) - K_1} \ ,$$

and the result for $I\!I\!I(\tau, \tau)$ follows. □

The previous proposition gives us informations about $\tilde{\nabla}$. Call K_I the curvature of I and K_e the extrinsic curvature of the immersion. Then:

Corollary 2.4. *$\tilde{\nabla}$ is a connection compatible with $I\!I\!I$, its torsion τ is bounded (for $I\!I\!I$), at $s \in S$, by $\tau_0(\phi(s))$ (where τ_0 comes from (1)), and its curvature is:*

$$\tilde{K} = \frac{K_I}{K_e} \ ,$$

with:

$$0 < K_4 \leq \tilde{K} \leq K_5 \ ,$$

where:

$$K_5 = 1 \ \textit{if} \ K_2 \geq 0 \quad , \quad K_5 = \frac{K_1}{K_1 - K_2} \ \textit{if} \ K_2 \leq 0$$

$$K_4 = 1 \ \textit{if} \ K_3 \leq 0 \quad , \quad K_4 = \frac{K_1}{K_1 - K_3} \ \textit{if} \ K_3 \geq 0 \ .$$

Proof. We only have to prove the second assertion, concerning the curvature. Let dv_I and $dv_{I\!I\!I}$ be the area elements associated to the metrics I and $I\!I\!I$ on Σ. By the Gauss formula (1):

$$dv_{I\!I\!I} = K_e dv_I \ .$$

Let (e_1, e_2) be an orthonormal moving frame on (Σ, I), and let ω be its connection 1-form, that is:

$$\omega(x) := I(\nabla_x e_1, e_2) = -I(\nabla_x e_2, e_1)$$

Then:

$$K_I dv_I = \Omega_I = -d\omega \ .$$

But $(B^{-1}e_1, B^{-1}e_2)$ is an orthonormal moving frame on $(\Sigma, I\!I\!I)$, and its connection 1-form $\omega_{I\!I\!I}$ is:

$$\omega_{I\!I\!I}(x) = I\!I\!I(\tilde{\nabla}_x(B^{-1}e_1, B^{-1}e_2) = \omega(x) \ .$$

Therefore:

$$\tilde{K} dv_{I\!I\!I} = \Omega_{I\!I\!I} = -d\omega_{I\!I\!I} = -d\omega = K dv_I \ .$$

Those equations give the relation we need between $\tilde{K}$, K_e and K_I.

The inequalities on $\tilde{K}$ are direct consequences of this formula, because:

$$\tilde{K} = \frac{K_I}{K_e} \leq \frac{K_I}{K_I - K_2}$$

Now the function: $x \mapsto x/(x-\alpha)$ has as derivative: $x \mapsto -\alpha/(x-\alpha)^2$, so its increasing for $\alpha \leq 0$ and decreasing for $\alpha \geq 0$; for $\alpha = K_2$ we find the upper bound on $\tilde{K}$ is obtained:

- if $K_2 \leq 0$, when $K_I \to K_1$, and it is $K_1/(K_1 - K_2)$;
- if $K_2 \geq 0$, when $K_I \to \infty$, and it is 1.

The same argument gives the lower bound for $\tilde{K}$, with K_2 remplaced by K_3. □

Lemma 1.1 is a direct consequence of proposition 2.3 and corollary 2.4.

We will now give two simple results which will be useful in the sequel. First, $\tilde{B} := B^{-1}$ satisfies on $(\Sigma, I\!I\!I, \tilde{\nabla})$ an equation similar to that satisfied by B on (Σ, I) but even simpler:

Proposition 2.5. *On $(\Sigma, I\!I\!I)$:*

$$d^{\tilde{\nabla}} \tilde{B} = 0$$

Proof. A direct computation shows that, for $s \in \Sigma$ and $X, Y \in T_s\Sigma$:

$$\begin{aligned} (d^{\tilde{\nabla}} \tilde{B})(X,Y) &= \tilde{\nabla}_X(\tilde{B}Y) - \tilde{\nabla}_Y(\tilde{B}X) - B^{-1}([X,Y]) \\ &= B^{-1}\nabla_X(BB^{-1}Y) - B^{-}\nabla_Y(BB^{-1}X) - B^{-1}[X,Y] \\ &= B^{-1}0 \ , \end{aligned}$$

because ∇ is torsion-free. □

We will now describe some properties of $\tilde{B}$ which will be useful later on. Remember that, since $\det(\tilde{B}) < 0$, there exist at each point of Σ two vectors U, V which have unit norm for $I\!I\!I$, and such that:

$$\begin{aligned} \tilde{B}U &= kJ_{I\!I\!I}U \\ \tilde{B}V &= -kJ_{I\!I\!I}V \ , \end{aligned}$$

where $J_{I\!I\!I}$ is the complex structure defined by $I\!I\!I$, and:

$$k = |\det(\tilde{B})|^{1/2} = (K_M - K_\Sigma)^{-1/2} \ .$$

U and V are a priori defined only up to their orientation, but, since we have supposed that Σ is contractible, we can decide that, in the remaining of this paper, U et V will be two globally defined vector fields, oriented so that $\angle(U,V) \in]0,\pi[$.

Remark 2.6. *The norms of U and V for I are at most $1/\sqrt{K_2 - K_1}$.*

Proof. By definition:

$$I(U,U) = I\!I\!I(\tilde{B}U, \tilde{B}U) = I\!I\!I(kJ_{I\!I\!I}U, kJ_{I\!I\!I}U) = k^2 = (K_M - K_\Sigma)^{-1} \leq (K_2 - K_1)^{-1}.$$

$\square$

The remainder of this section is dedicated to some elementary facts about the asymptotic curves of the immersion, as seen on $(\Sigma, I\!I\!I, \tilde{\nabla})$. Those curves have been well studied on (Σ, I); for instance, they have been used before [Efi64] in [Efi62] to prove that there exists a constant k such that, if a smooth, complete Riemannian surface S has uniformly negative curvature, and if the norm of the gradient of this curvature is bounded by k, then S has no isometric immersion into $\mathbf{R}^3$. But we only give here some details on the local behavior of asymptotic curves on $(\Sigma, I\!I\!I, \tilde{\nabla})$.

In all this paper, θ denotes the angle between U and V for $I\!I\!I$. As above, we suppose that $\theta \in]0, \pi[$. Note that θ is close to 0 (or to π) when the immersion ϕ is "degenerate": the mean curvature of ϕ is $\cot(\theta)(|\det(\tilde{B})|)^{-1/2}$.

Proposition 2.7. *At each point of Σ:*

$$\tilde{\nabla}_V U = -\frac{\sin(\theta)}{2}(U.\kappa + I\!I\!I(\tau, J_{I\!I\!I}U))J_{I\!I\!I}U \quad (1)$$

$$\tilde{\nabla}_U V = \frac{\sin(\theta)}{2}(V.\kappa + I\!I\!I(\tau, J_{I\!I\!I}V))J_{I\!I\!I}V \ , \quad (2)$$

with $\kappa = \ln(k^{-1}) = -\ln(k)$

Proof. From (1):

$$(d^{\tilde{\nabla}}\tilde{B})(U,V) = 0 \ ,$$

so, if $\omega_U := I\!I\!I(\tilde{\nabla}_U V, J_{I\!I\!I}V)$ and $\omega_V := I\!I\!I(\tilde{\nabla}_V U, J_{I\!I\!I}U)$:

$$\tilde{\nabla}_U(\tilde{B}V) - \tilde{\nabla}_V(\tilde{B}U) - \tilde{B}(\tilde{\nabla}_U V - \tilde{\nabla}_V U - \sin(\theta)\tau) = 0 \ ,$$

so that:

$$-\tilde{\nabla}_U(kJ_{I\!I\!I}V) - \tilde{\nabla}_V(kJ_{I\!I\!I}U) - \tilde{B}(\omega_U J_{I\!I\!I}V) + \tilde{B}(\omega_V J_{I\!I\!I}U) + \sin(\theta)\tilde{B}\tau = 0 \ .$$

But $\sin(\theta)J_{I\!I\!I}U = V - \cos(\theta)U$ and $\sin(\theta)J_{I\!I\!I}V = \cos(\theta)V - U$, and it follows that:

$$\omega_V U + \omega_U V + \left(-V.\kappa + \frac{\omega_U}{\sin(\theta)} - \frac{\omega_V \cos(\theta)}{\sin(\theta)}\right) J_{I\!I\!I}U +$$
$$+ \left(-U.\kappa + \frac{\omega_U \cos(\theta)}{\sin(\theta)} - \frac{\omega_V}{\sin(\theta)}\right) J_{I\!I\!I}V + \frac{\sin(\theta)}{k}\tilde{B}\tau = 0 \ .$$

Take the scalar product (for $I\!I\!I$) with U and then with V, and use the symmetry of $\tilde{B}$ with respect to $I\!I\!I$ to obtain the result. $\square$

We will use this proposition to show that U and V each behave well along the integral curves of the other. This will be used in section 4 to obtain a key technical lemma on asymptotic curves. Note that the hypothesis of theorem 0.2 on the gradient of the curvature appears only here.

Remember that, according to the hypothesis of theorem 0.2:

- There exists $c_\sigma > 0$ such that, for all $s \in \Sigma$ and all $x \in T_s\Sigma$:

$$\|x.K_\sigma\| \leq c_\sigma \|x\|_\sigma |K_\sigma|^{3/2} \ .$$

- There exists $c_\mu > 0$ such that, for all $m \in M$ and all $x \in T_m M$, for each 2-plane $P \in G^2_m M$:

$$|(\nabla^M_x K_\mu)(P)| \leq c_\mu \|x\| \ .$$

Then:

Corollary 2.8. *There exists $\tau_1 > 0$ (depending on $K_1, K_2, K_3, c_\sigma, c_\mu$ only) such that:*

$$\begin{aligned} \|\tilde{\nabla}_U V\|_{I\!I\!I} &\leq \tau_1|\sin(\theta)| & (3)\\ \|\tilde{\nabla}_V U\|_{I\!I\!I} &\leq \tau_1|\sin(\theta)| \ . & (4) \end{aligned}$$

Proof. According to the previous proposition:

$$\begin{aligned} \|\tilde{\nabla}_U V\|_{I\!I\!I} &\leq \left|\frac{\sin(\theta)}{2}\right| \left(\left|\frac{V.K_e}{2K_e}\right| + \|\tau\|_{I\!I\!I}\right) \\ &\leq \left|\frac{\sin(\theta)}{4}\right| \left(\left|\frac{V.K_\sigma}{K_e}\right| + \left|\frac{V.K_\mu}{K_e}\right| + 2\tau_0\right) \ . \end{aligned}$$

Let $x \in M$, call K_μ^x the restriction of K_μ to the Grassmannian of 2-planes in T_xM. A simple compactness argument shows that there exists a constant C_M (which does not depend on M) such that:

$$dK_\mu^x \leq C_M K_M^x \ ,$$

where K_M^x is the maximum of the sectional curvatures of M at x. Therefore, isolating in $V.K_\mu$ a part coming from the derivative of K_μ from another coming from the rotation of the tangent plane during a displacement in the direction of V shows that:

$$\|\tilde{\nabla}_U V\|_{I\!I\!I} \leq \left|\frac{\sin(\theta)}{4}\right| \left(\left|\frac{V.K_\sigma}{K_e}\right| + \left|\frac{(\tilde{\nabla}_{\phi_* V} K_\mu)(\phi_*(T_s\Sigma))}{K_e}\right| + \|V\|_{I\!I\!I} C_M \left|\frac{K_M}{K_e}\right| + 2\tau_0\right) \ ,$$

because the norm of the rotation of $\phi_* T_s\Sigma$ during displacements along Σ is measured by $I\!I\!I$. But $\|V\|_{I\!I\!I} = 1$ and $\|V\|_I = k = K_e^{-1/2}$, so:

$$\|\tilde{\nabla}_U V\|_{I\!I\!I} \leq \left|\frac{\sin(\theta)}{4}\right| \left(k^2|V.K_\sigma| + k^2|(\tilde{\nabla}_{\phi_* V} K_\mu)(\phi_*(T_s\Sigma)) + \left|\frac{C_M K_M}{K_e}\right| + 2\tau_0\right) \ ,$$

and, if k_M is the maximal possible value of k, i.e. $k_M = (K_2 - K_1^{-1/2}$:

$$\|\tilde{\nabla}_U V\|_{I\!I\!I} \leq \left|\frac{\sin(\theta)}{4}\right| \left(k_M^3 c_\sigma + k_M^3 c_\mu + k_M^2 C_M |K_3| + 2\tau_0\right) \ ,$$

whence the first result. The same computation with U and V interchanged gives the same bound for $\|\tilde{\nabla}_V U\|_{I\!I\!I}$. □

Lemma 1.2 is no more than a restatement of corollary 2.8.

3 Connections with bounded torsion

This section contains some simple technical propositions describing some properties of surfaces with metrics and compatible connections with bounded torsion.

First note that the Gauss-Bonnet theorem remains valid in this setting: if D is a compact, simply connected domain in Σ with smooth boundary, then the integral of the geodesic curvature (for $\tilde{\nabla}$) of ∂D is equal to 2π minus the integral of the curvature $\tilde{K}$ of $\tilde{\nabla}$ over D.

This is proved as follows. Let (X, Y) be an orthogonal moving frame on $D \setminus \{p\}$, where p is a point in D, with X tangent to ∂D and to the "circles" $\partial B(p, \epsilon)$ for ϵ small enough. Let ω the connection 1-form of (X, Y), and Ω its curvature 2-form. By definition of $\tilde{K}$:

$$\Omega = \tilde{K} dv \ ,$$

where dv is the area form of $I\!I\!I$; moreover:

$$\Omega = -d\omega \ ,$$

so

$$\int_D \Omega = -\int_{\partial M} \omega - \lim_{\epsilon\rightarrow 0}\int_{\partial B(p,\epsilon)} \omega \ .$$

Therefore, if κ is the geodesic curvature of ∂D:

$$\int_D \tilde{K} dv = -\int_{\partial D} \kappa ds + 2\pi \ .$$

This theorem of course remains true if ∂D is only piecewise smooth, with the adequate contributions from the singular points.

We now describe some properties of geodesics which ressemble those for Jacobi fields along geodesics when the connection has no torsion. But the torsion comes into the equations so that the usual equalities are replaced by inequalities.

Let $(g_s)_{s\in[0,1]}$ be a family of $\tilde{\nabla}$-geodesic, $g_s : [0,L] \rightarrow \Sigma$, parametrized at unit speed. For each $s \in [0,1]$ and each $t \in [0,L]$, we let $g' := \partial g_s(t)/\partial t$ and $\overset{\bullet}{g} := \partial g_s(t)/\partial s$. For $s = 0$, $\overset{\bullet}{g}$ is a kind of Jacobi field along g_0, and we can call x and y the functions from $[0,L]$ to $\mathbf{R}$ such that, for $s = 0$:

$$\overset{\bullet}{g} = xg' + yJ_{I\!I\!I} g' \ .$$

We also call $\tau_x(t) := I\!I\!I(\tau, g'_s(t))$ and $\tau_y := I\!I\!I(\tau, J_{I\!I\!I} g'_s(t))$.

Proposition 3.1. *x and y are solutions of:*

$$x' = y\tau_x$$

$$y'' = -\tilde{K}y + (y\tau_y)' \ .$$

Proof. By definition of g' and $\overset{\bullet}{g}$, $[g', \overset{\bullet}{g}] = 0$, so that, by definition of the torsion:

$$\tilde{\nabla}_{\overset{\bullet}{g}} g' = \tilde{\nabla}_{g'} \overset{\bullet}{g} - \tau(g', \overset{\bullet}{g}) \ .$$

Taking the scalar product with g' and using the fact that the (g_s) are parametrized at unit speed shows that:

$$0 = \overset{\bullet}{g} . I\!I\!I(g', g') = 2I\!I\!I(\tilde{\nabla}_{g'} \overset{\bullet}{g} - \tau(g', \overset{\bullet}{g}), g') \ .$$

Therefore:

$$g'.I\!I\!I(\overset{\bullet}{g}, g') - I\!I\!I(\tau(g', \overset{\bullet}{g}), g') = 0$$

and we obtain the first equation.

Coming back to equation (5), we see that:

$$\begin{aligned}
\tilde{\nabla}_{g'}\tilde{\nabla}_{g'} \overset{\bullet}{g} &= \tilde{\nabla}_{g'}\tilde{\nabla}_{\overset{\bullet}{g}} g' + \tilde{\nabla}_{g'}(\tau(g', \overset{\bullet}{g})) \\
&= R_{g',\overset{\bullet}{g}} g' + \tilde{\nabla}_{\overset{\bullet}{g}}\tilde{\nabla}_{g'} g' + \tilde{\nabla}_{g'}(\tau(g', \overset{\bullet}{g})) \\
&= -\tilde{K}yJ_{I\!I\!I} g' + \tilde{\nabla}_{g'}(y\tau_x g' + y\tau_y J_{I\!I\!I} g') \\
&= (y\tau_x)' g' + (-\tilde{K}y + (y\tau_y)')J_{I\!I\!I} g' \ ,
\end{aligned}$$

and the second equations follows (as well as the derivative of the first). □

Corollary 3.2. *There exists $t_g > 0$, depending on K_4, K_5 and τ_0, such that, if $x(0) = y(0) = 0$, then, for all $t \in [0, t_g]$:*

$$\frac{y'(0)t}{2} \le y(t) \le 2y'(0)t$$

$$|x(t)| \le \tau_0 y'(0)t^2 \ .$$

Proof. Integrating (5) shows that, for $t \in [0, L]$:

$$y'(t) - y'(0) = \int_0^t -K(s)y(s)ds + y(t)\tau_y(t) \ ,$$

so that:

$$-\tau_0 y(t) - K_5 \int_0^t y(s)ds \le y'(t) - y'(0) \le \tau_0 y(t) - K_4 \int_0^t y(s)ds \ .$$

Let:

$$t_1 := \inf\{t \ge 0 \mid y(t) \notin [y'(0)t/2, 2y'(0)t]\} \ .$$

For $t \le t_1$:

$$y'(0)(1 - 2\tau_0 t - K_5 t^2) \le y'(t) \le y'(0)(1 + 2\tau_0 t - K_4 t^2/4) \ .$$

Thus there exists $t_g > 0$ such that, if $t_1 < t_g$, then:

$$\frac{y'(0)t}{2} \le y(t) \le 2y'(0)t \ ,$$

which contradicts the definition of t_1. So $t_1 \ge t_g$, and equation (5) follows. (5) is a direct consequence using (5). ☐

Corollary 3.3. *If $x \in \Sigma$ and $v \in T_x\Sigma$ is a vector of norm at most t_g at which the exponential at x for $\tilde{\nabla}$, $\exp_x^{\tilde{\nabla}}$, is defined, then $\exp_x^{\tilde{\nabla}}$ is a local diffeomorphism at v.*

Proof. Let:

$$v' := \Pi(\exp_x^{\tilde{\nabla}}([0,1]v), v) \ .$$

Equation (5) shows that:

$$I\!I\!I((d_v \exp_x^{\tilde{\nabla}})(J_{I\!I\!I} v), J_{I\!I\!I} v') \ne 0 \ ,$$

while it is easy to check that:

$$(d_v \exp_x^{\tilde{\nabla}})(v) = v' \ ,$$

because this corresponds to a change in the parametrization of the geodesic starting at x in the direction of v. ☐

Corollary 3.4. *Let $\Omega \subset \Sigma$ be an open subset with locally convex boundary, $\overline{\Omega} \subset \Sigma$. For any $x, y \in \Omega$ with $d_{I\!I\!I}(x, y) \le t_g$, there exists a unique $\tilde{\nabla}$-geodesic of length $d_{I\!I\!I}(x, y)$ between x and y.*

Proof. Let Ω' be the inverse image of Ω by the restriction of $\exp_x^{\tilde{\nabla}}$ to the ball of radius t_g. By the previous corollary and the local convexity of Ω, the restriction of $\exp_x^{\tilde{\nabla}}$ is a diffeomorphism onto its image. ☐

Here is another elementary corollary of proposition 3.1.

Corollary 3.5. *For all $\epsilon > 0$, there exists $\alpha > 0$ such that , if $L \le \alpha$ and $y'(0) = x(0) = 0$, then:*

$$\forall t \in [0, L], |y(t) - y(0)| \le \epsilon$$

$$\left| x(L) - y(0) \int_0^L \tau_y(s)ds \right| \le \epsilon L$$

Proof. (5) is a simple consequence of (5), and (5) then follows from (5). ☐

We can now consider a family of geodesic rays starting from a given point, and describe how they behave relative to one another. Let $(g_\theta)_{\theta\in[0,\theta_0]}$ be a family of maximal rays, with $g_\theta : [0, L_\theta) \to \Sigma$, $L_\theta \in \mathbf{R}_+^* \cup \{\infty\}$, and with $g_\theta(0) = g_0(0)$ and $\angle(g_0'(0), g_\theta'(0)) = \theta$ for each $\theta \in [0, \theta_0]$.

For $s \in [0, L_0)$, let n_s be the maximal geodesic ray with $n_s(0) = g_0(s)$ and $n_s'(0) = J_{I\!I\!I} g_0'(s)$. Choose $s_1 > 0$ and $\theta_1 > 0$, and suppose that there is no θ, t_0, s, u_0 with $s \le s_1$ and $\theta \le \theta_1$ such that:

$$\lim_{t\to t_0} g_\theta(t) = \lim_{u\to u_0} n_s(u) \in \partial_{I\!I\!I}\Sigma \ .$$

Then:

Proposition 3.6. *There exists a constant $S > 0$ and, for each $\epsilon > 0$ small enough and each $s_1 > 0$, there exists $\Theta(\epsilon, s_1) > 0$ (both also depending on τ_0, K_4, K_5) such that, if $s \le s_1$ and $\theta \le \Theta(\epsilon, s_1)$, then:*

1. *g_θ intersects n_s at a point $n_s(u_\theta(s))$ (with $g_\theta \cap n_s([0, u_\theta(s))) = \emptyset$);*
2. *the restriction of $|u_\theta|$ to $[0, s]$ remains bounded by ϵ;*
3. *if $s \ge S$, there exists $s' \in [0, s]$ such that $u_\theta(s') = -\epsilon\theta$.*

Proof. Let u_M be a small real number; we will see later how small u_M has to be. For $\theta \in [0, \theta_1]$, let:

$$\alpha_\theta(s) := \angle(-J_{I\!I\!I} n_s'(u_\theta(s)), g_\theta') \ .$$

Then define:

$$s_\theta := \sup\{s \in \mathbf{R}_+ \mid \forall s' \in [0, s], |u_\theta(s')| \le u_M \text{ and } |\alpha_\theta(s')| \le u_M\} \ .$$

For $s \in [0, s_\theta]$, apply the Gauss-Bonnet theorem to an infinitesimal strip bounded by $g_0([s, s+ds])$, $n_s([0, u_\theta(s)])$, $n_{s+ds}([0, u_\theta(s+ds)])$ and g_θ. This shows that:

$$\alpha_\theta'(s) = -\int_0^{u_\theta(s)} \tilde{K}(n_s(t)) \left\| n_s'(t) \wedge \frac{\partial}{\partial s} n_s(t) \right\| dt \ ,$$

so that:

$$\alpha_\theta'(s) = -k(s) u_\theta(s) \ ,$$

where $k(s) \in [K_4 - \epsilon, K_5 + \epsilon]$ if u_M is small enough (this last step uses corollary 3.5 applied to the family (n_s)).

Again by corollary 3.5, it is not hard to check that, again for u_M small enough:

$$\left\| \frac{\partial}{\partial s} n_s(u_\theta(s)) + J_{I\!I\!I} n_s'(u_\theta(s)) \right\| \le \frac{\epsilon}{4} \ .$$

Thus, with (5):

$$\left(1 - \frac{\epsilon}{4}\right) \sin\alpha_\theta(s) + (1-\epsilon) \int_0^{u_\theta(s)} \tau(-J_{I\!I\!I} n_s'(t)) dt \le u_\theta'(s) \le$$

$$\le \left(1 + \frac{\epsilon}{4}\right) \sin\alpha_\theta(s) + (1+\epsilon) \int_0^{u_\theta(s)} \tau(-J_{I\!I\!I} n_s'(t)) dt \ .$$

This can be written, for u_M small enough, as:

$$u_\theta'(s) = \lambda(s)\alpha_\theta(s) + \tau(s) u_\theta(s) \ ,$$

with:

$$|\lambda(s) - 1| \le \epsilon, \quad |\tau(s)| \le \tau_0(1+\epsilon) \ .$$

Let:

$$X(s) := \begin{pmatrix} u_\theta(s) \\ \alpha_\theta(s) \end{pmatrix} \ .$$

Then:

$$X'(s) = m(s)X(s) \ ,$$

with:

$$m(s) := \begin{pmatrix} \tau(s) & \lambda(s) \\ -k(s) & 0 \end{pmatrix} \ .$$

Thus, by integration:

$$X(s) = \exp(sM(s))X(0) \ ,$$

where:

$$M(s) := \begin{pmatrix} T(s) & \Lambda(s) \\ -K(s) & 0 \end{pmatrix} \ ,$$

with:

$$|T(s)| \leq (1+\epsilon)\tau_0, \ |\Lambda(s) - 1| \leq \epsilon, \ K_4 - \epsilon \leq K(s) \leq K_5 + \epsilon \ .$$

The eigenvalues of $M(s)$ are the roots of:

$$X(X - T(s)) + \Lambda(s)K(s) = 0 \ .$$

If ϵ is so small that $4(K_4 - \epsilon)(1-\epsilon) > (1+\epsilon)^2\tau_0$, those roots can be written as $\alpha \pm i\beta$, where:

$$|\alpha| = \frac{T(s)}{2} \leq \frac{(1+\epsilon)\tau_0}{2}, \ |\beta| \geq \frac{\sqrt{4(K_4-\epsilon)(1-\epsilon) - (1+\epsilon)^2\tau_0^2}}{2} \ .$$

Therefore, in a well chosen frame, the orbits of $X(s)$ are "spirals" around 0, with an angular speed which is bounded from below. This already proves, with the upper bound on α, that, if θ is smaller than some $\Theta(\epsilon, s)$, then $s_\theta \geq s$, so that u_M is not reached and the computations above hold on all of $[0, s]$. This proves point (2).

Moreover, the trajectories $(X(s'))_{s' \in [0,s]}$ can not remain in a half-plane, so that u_θ has to become negative after a time which is bounded in term of β (which itself is bounded from below). This leads to point (3) of the proposition. □

Finally, the same kind of argument will show the following similar proposition, which deals with convex curves instead of geodesics. The proof is similar to the one we have just finished, so it is described somewhat faster.

Proposition 3.7. *Let S be a convex domain in Σ, with boundary ∂S containing as connected components two complete curves γ and $\tilde\gamma$. Suppose that $K_4 > \tau_0^2/4$. Then $d_{III}(\gamma, \tilde\gamma) > 0$.*

Proof. If γ or $\tilde\gamma$ is compact, the result is obvious, so we suppose here that neither γ nor $\tilde\gamma$ is compact. The proof is by contradiction, so we suppose that $d_{III}(\gamma, \tilde\gamma) = 0$.

First note that a rather direct smoothing argument shows that, for any $\epsilon_r > 0$, there are smooth curves $\gamma_r, \tilde\gamma_r : \mathbf{R} \to \Sigma$ such that:

- $(\partial S \setminus (\gamma \cup \tilde\gamma)) \cup (\gamma_r \cup \tilde\gamma_r)$ bounds a connected closed set S_r which contains S;
- for each $s \in \mathbf{R}$, the curvatures $\kappa(t)$ and $\tilde\kappa(t)$ of γ_r at $\gamma_r(t)$ and of $\tilde\gamma_r$ at $\tilde\gamma_r(t)$ respectively are bounded by:
$$\kappa(t) \geq -\epsilon d_{III}(\gamma_r(t), \tilde\gamma_r) \ , \quad \tilde\kappa(t) \geq -\epsilon d_{III}(\tilde\gamma_r(t), \gamma_r) \ ,$$
where both curvatures are with respect to the normal oriented towards the interior of S';
- $\liminf_{t \to \infty} d_{III}(\gamma_r(t), \tilde\gamma_r) = 0$.

For $s \in \mathbf{R}$, let:

$$d(s) = d_{III}(\gamma_r(s), \tilde\gamma_r) \ .$$

Thus d is not bounded away from 0 near $+\infty$.

Choose $\epsilon > 0$. There exists $s_0 \in \mathbf{R}$ with

$$d(s_0) \leq \epsilon, \ d'(s_0) \leq \epsilon \ .$$

If ϵ is small enough, it is not difficult to show, using 3.4, that there exists a $\tilde{\nabla}$-geodesic n_{s_0} connecting $\gamma_r(s_0)$ to $\tilde{\gamma}_r$, of length at most 2ϵ, orthogonal to $\tilde{\gamma}_r$. For $s > s_0$, let n_s be the maximal $\tilde{\nabla}$-geodesic starting at $\gamma_r(s)$ with speed equal to the parallel transport of $n_s'(0)$ at $\gamma_r(s)$ along γ_r.

Let $r(s)$ be the distance along n_s between $\gamma_r(s)$ and the first intersection of $n(s)$ with $\tilde{\gamma}_r$, $\beta(s)$ the angle between $-J_{I\!I\!I} n_s'(0)$ and $\gamma_r'(s)$, α the angle between $-J_{I\!I\!I} n_s'(r(s))$ and $\tilde{\gamma}_r'$. By construction, $\alpha(s_0) = 0$, while, by corollary 3.5 and (5), $\beta(s_0)$ is small. Let u_M be again a small real number, for which precisions will come later. Define:

$$s_M := \sup\{s \geq s_0 \mid \forall s' \in [s_0, s], |u(s')| \leq u_M \text{ and } |\alpha(s')| \leq u_M \text{ and } |\beta(s')| \leq u_M\} \ .$$

The definition of β and the “almost” convexity of γ_r show that $\beta'(s) \geq -2\epsilon u(s)$, while the same application of the Gauss-Bonnet theorem as the one leading to (5) shows again that $\alpha'(s) = -k(s)u(s)$, but with only $k(s) \geq K_4 - 2\epsilon$, while the upper bound is lost because $\tilde{\gamma}_r$ is only “almost convex” instead of geodesic.

Moreover, the same argument as the one leading to (5) shows that:

$$u'(s) = \lambda(s)(\alpha(s) - \beta(s)) + \tau(s)u(s) \ ,$$

again with:

$$|\lambda(s) - 1| \leq \epsilon, \quad |\tau(s)| \leq \tau_0(1 + \epsilon) \ .$$

The rest of the proof can now be done just as in the proof of proposition 3.6, with α_θ replaced by $\alpha - \beta$, to obtain that there exists $S > 0$ (depending on K_4 and τ_0) such that:

- either there exists $s \in [s_0, s_0 + S]$ such that $u(s) = 0$, and this proves the proposition;
- or $s_M < s_0 + S$, and in this case the upper bound on the norm of X shows that, if ϵ has been chosen small enough, then either $\alpha(s) = -u_M$ or $\beta(s) = u_M$.

But then, again for ϵ small enough, it is not difficult to show that there exists $S' > 0$ such that there exists $s \in (s_M, s_M + S')$ such that $u(s) = 0$, so that the proposition holds also in that case. □

4 Asymptotic curves

This section contains the proof of lemma 4.3, a technical statement which will have a central role later on. This lemma, along with its proof, is similar to a lemma from [Sch99], but more detailed estimates are necessary here. First, we introduce a simple notation. It is written for an integral curve of U, but the analog for an integral curve of V should be obvious.

Definition 4.1. *Let $\gamma : [0, L] \to \Sigma$ be an integral curve of U or V. Then:*

$$\delta_\gamma := \pi + \inf_{t \in [0,L]} \theta(\gamma(t)) - \sup_{t \in [0,L]} \theta(\gamma(t)) \ ,$$

and:

$$\sigma_\gamma := \int_0^L \sin(\theta(\gamma(s)))ds \ .$$

Thus $\delta_\gamma \in (0, \pi)$; heuristically, because of (3) and (4), δ_γ is small when γ has a segment which looks like a closed loop. The following definition is very natural:

Definition 4.2. *Let $\epsilon > 0$. A curve $c : [0, L] \to \Sigma$ is an ϵ-quasi-geodesic if, for each $s \in [0, L]$, the absolute value of the angle between $c'(s)$ and $\Pi(c_{|[0,s]}; c'(0))$ is at most ϵ.*

Lemma 4.3. *There exists $T_0 > 0, C_0 > 0$ and $\epsilon_0 > 0$ as follows. Let g be an integral curve of U of length $L_g \leq T_0$, with $\epsilon := \max(\delta_g, \sigma_g) \leq \epsilon_0$. Let $h_u : [-T_0, T_0] \to \Sigma$ be the integral curve of V with $h_u(0) = g(u)$. Then, for any $u \in [0, L_g]$, h_u is a $C_0\epsilon$-quasi-geodesic.*

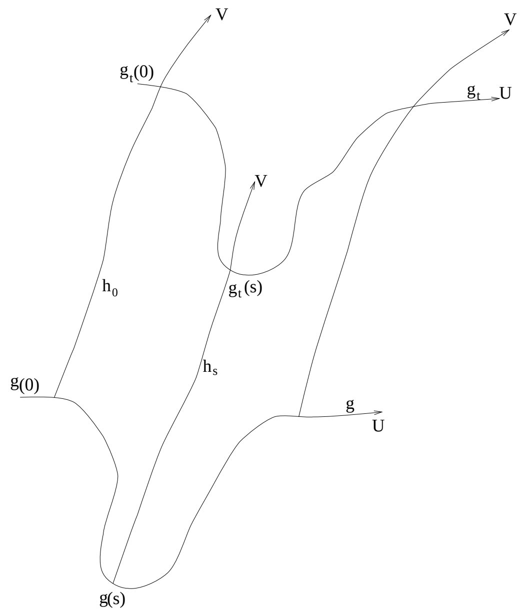

Figure 4.1

More could actually be said: the curve g "propagates" along the flow of V, that is, under this flow V, the integral curves of U corresponding (in some natural sense) to g still have very small values of δ and of σ. This should be clear from the proof, although we do not elaborate on it since it is not used later on.

The proof of this lemma rests on the following:

Proposition 4.4. *There exists $L_M > 0$ and continuous functions $\phi, \Phi : [0, L_M] \times [0, L_M] \to \mathbf{R}_+$ such that, for all $L \in [0, L_M]$, $\phi(0, L) = 0$, and $\Phi(0, L) = \Phi(L, 0) = 0$, with the following properties. If $g : [0, L_0] \to \Sigma$ and $\overline{g} : [0, L_1] \to \Sigma$ are integral curves of U, if $h : [0, L'_0] \to \Sigma$ and $\overline{h} : [0, L'_1] \to \Sigma$ are integral curves of V, with $g(0) = h(0)$, $g(L_0) = \overline{h}(0)$, $\overline{g}(0) = h(L'_0)$ and $\overline{g}(L_1) = \overline{h}(L'_1)$, and if $L_0 \leq L_M$ and $L'_0 \leq L_M$, then $L_1 \leq \phi(L_0, L'_0)$, $L'_1 \leq \phi(L'_0, L_0)$, and the area of the domain bounded by $g, \overline{g}, h$ and $\overline{h}$ is at most $\Phi(L_0, L'_0)$.*

Proof. Let $u \in [0, L_0]$ and $v \in [0, L'_0]$. We can suppose that the integral curve of V starting at $g(u)$ meets the integral curve of U starting from $h(v)$: otherwise, the proposition would fail slightly before the first value of u such that the intersection does not exist, because then the length of both $\overline{g}$ and $\overline{h}$ would go to infinity. We call $g_v(u)$ the intersection of the integral curve of V starting at $g(u)$ with the integral curve of U starting from $h(v)$; this intersection has to be unique because Σ is simply connected and U and V are transverse.

Let $\partial_u = \partial/\partial u, \partial_v = \partial/\partial v$. Then:

$$\partial_v g_v(u) = \alpha(u, v)V \ , \ \ \partial_u g_v(u) = \beta(u, v)U \ .$$

By definition of τ:

$$\tilde{\nabla}_{\alpha V}(\beta U) - \tilde{\nabla}_{\beta U}(\alpha V) - [\alpha V, \beta U] = \tau(\alpha V, \beta U) \ ,$$

so:

$$(\partial_v \beta)U + \alpha\beta\tilde{\nabla}_V U - (\partial_u \alpha)V - \alpha\beta\tilde{\nabla}_U V = -\alpha\beta\sin(\theta)\tau \ .$$

Take the scalar product (for $I\!I\!I$) of this equation with $J_{I\!I\!I}U$ to obtain that:

$$-\sin(\theta)\partial_u\alpha + \alpha\beta(\langle\tilde{\nabla}_V U, J_{I\!I\!I}U\rangle_{I\!I\!I} - \langle\tilde{\nabla}_U V, J_{I\!I\!I}U\rangle_{I\!I\!I}) = -\alpha\beta\sin(\theta)\langle\tau, J_{I\!I\!I}U\rangle_{I\!I\!I} \ ,$$

which shows, along with lemmas 1.1 and 1.2, that:

$$|\partial_u\alpha| \leq (\tau_0 + 2\tau_1)|\alpha\beta| \ .$$

The same proof can be used to show also that:

$$|\partial_v \beta| \leq (\tau_0 + 2\tau_1)|\alpha\beta| \ .$$

In other terms:

$$|U.\alpha| \leq (\tau_0 + 2\tau_1)\alpha \ , \quad |V.\beta| \leq (\tau_0 + 2\tau_1)\beta \ .$$

Moreover, $\alpha(0, v) = 1$ and $\beta(u, 0) = 1$.

Integrate (5) over g_v to obtain that:

$$\alpha(u, v) \leq \exp((\tau_0 + 2\tau_1)L(g_v)) \ .$$

Using (5) again leads to:

$$\begin{aligned}
\frac{d}{dv}L(g_v) &= \frac{d}{dv}\int_0^{L_0} \beta(u, v)du \\
&= \int_0^{L_0} \partial_v \beta du \\
&\leq \int_0^{L_0} (\tau_0 + 2\tau_1)\alpha(u, v)\beta(u, v)du \\
&\leq (\tau_0 + 2\tau_1)\left(\sup_{u\in[0,L_0]} \alpha(u, v)\right)\int_0^{L_0} \beta(u, v)du \ ,
\end{aligned}$$

so:

$$\frac{d}{dv}L(g_v) \leq (\tau_0 + 2\tau_1)\exp((\tau_0 + 2\tau_1)L(g_v))L(g_v)$$

Now integrate this equation to obtain the required upper bound on L_1; the upper bound on L'_1 is obtained in the same way, exchanging u and v. Finally, the upper bound on the area comes from the upper bounds on $L(g_v)$ and on $\sup_{u\in[0,L_0]} \alpha(u, v)$ which we have found. □

Corollary 4.5. *Let $x, y, z \in \Sigma$ be such that there exists an integral curve of U (or of $-U$) of length at most L_M going from x to y, and an integral curve of V (or of $-V$) of length at most L_M going from x to z. Then the integral curve of U through z meets the integral curve of V through y.*

Proof. The intersections between those integral curves remain at bounded distance as long as the lengths of the integral curves of U and V going from x to y and to z remain below L_M, and the integral curves of U and of V do not meet $\partial_{I\!I\!I}\Sigma$ because (Σ, I) is complete (cf. section 2). □

Proof of lemma 4.3. Choose $u \in [0, L_g]$ and $t \in [-2T_0, 2T_0]$. From corollary 4.5, if T_0 and L_g are below a fixed constant, then the integral curve of U starting at $h_0(t)$ meets the integral curve of V starting at $g(u)$. Let $g_t(u)$ be their intersection, and $A(u, t)$ be the area of the domain in Σ bounded by g, $g_{[0,t]}(L_g)$, g_t, $g_{[0,t]}(0)$. By proposition 4.4, there exists some $T_1 > 0$ such that, if $L_g \leq T_1$ and $2T_0 \leq T_1$, then, for all $u \in [0, L_g]$:

$$\left\|\frac{\partial g_t(u)}{\partial u}\right\| \leq 2 \ , \quad \left\|\frac{\partial g_t(u)}{\partial t}\right\| \leq 2 \ .$$

By definition of δ_g, there exists $u_0, u_1 \in [0, L_g]$ such that $\theta(g(u_0)) - \theta(g(u_1)) \geq \pi - \delta_g$. Then $\theta(g(u_0)) \geq \pi - \delta_g$ and $\theta(g(u_1)) \leq \delta_g$. To simplify the notations a little, we suppose that $u_0 < u_1$. For each $t \in [-2T_0, 2T_0]$, let:

$$\theta_t := \max(\pi - \theta(g_t(u_0)), \theta(g_t(u_1))) \ .$$

We now suppose that $t \in [-2T_0, 2T_0]$ is such that, for all $s \in [0, t]$, $\theta_s \leq 2C_1\epsilon$ and $\sigma_{g_s} \leq 2C_1\epsilon$, for some constant C_1 on which more details will be given later. We will show that, if T_0 is small enough, then this implies that $\theta_t \leq C_1\epsilon$ and $\sigma_{g_t} \leq C_1\epsilon$, so that the same bounds apply for all $t \in [-2T_0, 2T_0]$.

Let C_t be the closed curve $g_0([u_0, u_1]) \cup g_{[0,t]}(u_1) \cup g_t([u_0, u_1]) \cup g_{[0,t]}(u_0)$. Let W be the vector field on C_t equal to V over $g_0([u_0, u_1]) \cup g_t([u_0, u_1])$ and to U over $g_{[0,t]}(u_1) \cup g_{[0,t]}(u_0)$. According to the Gauss-Bonnet theorem, the total rotation of W on C_t (i.e. the integral of $\langle\tilde{\nabla}W, JW\rangle$ plus the terms corresponding to the vertices) is $-K_t$, where K_t is the integral of $\tilde{K}$ on the interior of C_t. But :

- The terms corresponding to $g_0([u_0, u_1])$ and to $g_t([u_0, u_1])$ are bounded because of (3):

$$\int_{g_0([u_0,u_1])} \|\tilde{\nabla}_U V\| \leq \int_{u_0}^{u_1} \tau_1 \sin\theta(g_0(u))du \leq \tau_1 \sigma_{g_0} \leq 2\tau_1 C_1 \epsilon \ ,$$

$$\int_{g_t([u_0,u_1])} \|\tilde{\nabla}_U V\| \leq \tau_1 \sigma_{g_t} \leq 2\tau_1 C_1 \epsilon \ .$$

- the terms corresponding to $g_{[0,t]}(u_1)$ and $g_{[0,t]}(u_0)$ are bounded because of (4):

$$\begin{aligned} \int_{g_{[0,t]}(u_0)} \|\tilde{\nabla}_V U\| &\leq \int_0^t \tau_1 \left\| \frac{\partial g_s(u_0)}{\partial s} \right\| \sin\theta(g_s(u_0))ds \\ &\leq \int_0^t 2\tau_1 \theta_s ds \\ &\leq 4\tau_1 C_1 \epsilon t \ . \end{aligned}$$

- K_t is bounded by:

$$\begin{aligned} K_t &\leq K_5 \int_{s=0}^t \int_{u=u_0}^{u_1} \left\| \frac{\partial g_s(u)}{\partial s} \right\| \left\| \frac{\partial g_s(u)}{\partial u} \right\| \sin\theta(g_s(u))duds \\ &\leq K_5 \int_{s=0}^t \int_{u=u_0}^{u_1} 4 \sin\theta(g_s(u))duds \\ &\leq 4K_5 \int_{s=0}^t \sigma_{g_s} ds \ , \end{aligned}$$

  so that:

$$K_t \leq 8K_5 C_1 \epsilon t \ .$$

Therefore:

$$|\theta(g_0(u_0)) - \theta(g_0(u_1)) + \theta(g_t(u_1)) - \theta(g_t(u_0))| \leq 4\tau_1 C_1 \epsilon + 8\tau_1 C_1 \epsilon t + 8K_5 C_1 \epsilon t \ .$$

Thus:

$$|(\pi - \theta(g_t(u_0))) + \theta(g_t(u_1))| \leq |(\pi - \theta(g_0(u_0))) + \theta(g_0(u_1))| + 4\tau_1 C_1 \epsilon(1 + 2t) + 8K_5 C_1 \epsilon t \ ,$$

so that:

$$|(\pi - \theta(g_t(u_0))) + \theta(g_t(u_1))| \leq \delta_g + 4\tau_1 C_1 \epsilon(1 + 2t) + 8K_5 C_1 \epsilon t \ .$$

This already shows that, if T_0 is such that:

$$1 + 4\tau_1 C_1(1 + 2T_0) + 8K_5 C_1 T_0 \leq C_1 \ ,$$

then $\theta_t \leq C_1\epsilon$. It remains to show that $\sigma_{g_t} \leq C_1\epsilon$.

Equation (4) shows that:

$$\begin{aligned} |\angle(U(g_t(u_0)), \Pi(g_{[0,t]}(u_0); U(g_0(u_0))))| &\leq \tau_1 \int_{g_{[0,t]}(u_0)} \sin\theta(g_s(u_0))ds \\ &\leq 2\tau_1 \int_0^t \theta_s ds \\ &\leq 2\tau_1 C_1 \epsilon t \ , \end{aligned}$$

and, since $\theta_t \leq C_1\epsilon$ and $\theta_0 \leq \epsilon$:

$$|\angle(V(g_t(u_0)), \Pi(g_{[0,t]}(u_0); V(g_0(u_0))))| \leq 2\tau_1 C_1 \epsilon t + (C_1 + 1)\epsilon \ .$$

On the other hand, (3) shows that, for all $u \in [0, L_g]$:

$$|\angle(V(g_t(u)), \Pi(g_t([u_0, u]); V(g_t(u_0))))| \leq \tau_1 \sigma_t \leq 2\tau_1 C_1 \epsilon \ ,$$

while, for the same reason:

$$|\angle(V(g_0(u)), \Pi(g_0([u_0, u]); V(g_0(u_0))))| \leq 2\tau_1 C_1 \epsilon \ .$$

Finally, the same argument as in the proof of (5) above shows that the integral of $\tilde{K}$ on the domain bounded by $g_0([u_0, u])$, $g_{[0,t]}(u)$, $g_t([u_0, u])$ and $g_{[0,t]}(u_0)$ is at most $8K_5 C_1 \epsilon t$. The Gauss-Bonnet theorem, applied to this domain, therefore indicates that:

$$|\angle(V(g_t(u)), \Pi(g_{[0,t]}(u); V(g_0(u))))| \leq 2\tau_1 C_1 \epsilon t + (C_1 + 1)\epsilon + 4\tau_1 C_1 \epsilon + 8K_5 C_1 \epsilon t \ ,$$

so that:

$$|\angle(V(g_t(u)), \Pi(g_{[0,t]}(u); V(g_0(u))))| \leq 8C_1 \epsilon t(\tau_1 + K_5) + (C_1 + 1)\epsilon \ .$$

Moreover, by (4):

$$\begin{aligned} |\angle(U(g_t(u)), \Pi(g_{[0,t]}(u); U(g_0(u))))| &\leq \int_{g_{[0,t]}(u)} \|\tilde{\nabla}_V U\| \\ &\leq \int_0^t 2\tau_1 \sin\theta(g_s(u)) ds \ , \end{aligned}$$

so that:

$$\begin{aligned} \int_0^{L_g} |\angle(U(g_t(u)), \Pi(g_{[0,t]}(u); U(g_0(u))))| du &\leq 2\int_0^{L_g} \int_0^t 2\tau_1 \sin\theta(g_s(u)) ds du \\ &\leq 4\tau_1 \int_0^t \sigma_{g_s} ds \\ &\leq 8C_1 \tau_1 \epsilon t \ . \end{aligned}$$

But the definition of σ_{g_t} shows that:

$$\sigma_{g_t} \leq \int_0^{L_g} \sin\theta(g_0(u)) + |\angle(U(g_t(u)), \Pi(g_{[0,t]}(u); U(g_0(u))))| + \\ + |\angle(V(g_t(u)), \Pi(g_{[0,t]}(u); V(g_0(u))))| du \ ,$$

so by (5):

$$\sigma_{g_t} \leq \sigma_{g_0} + 16C_1 \tau_1 \epsilon t + 2L_g(8C_1 t(\tau_1 + K_5) + C_1 + 1)\epsilon \ .$$

But $|t| \leq 2T_0$, $L_g \leq T_0$ and $\sigma_{g_0} \leq 2C_1 \epsilon$, so it is clear that there exists C_1 such that, if T_0 is small enough, $\sigma_{g_t} \leq C_1 \epsilon$.

Using (5) once more then proves the lemma. □

5 Concave points

We now turn to the proof of lemma 1.7, which we recall here for the reader's convenience.

Lemma 1.7. *Under the hypothesis of theorem 0.2, $\partial_{I\!I\!I} \Sigma$ has no concave point.*

When $\phi : [-d, d] \times [0, d] \rightarrow \Sigma$ is a (k, C)-concave map, and when $d' \leq d$, we call $\phi_{d'}$ the restriction of ϕ to $[-d', d'] \times [0, d']$; it is again a (k, C)-concave map. We also call:

$$\partial_R \phi := \phi(\partial([-d, d] \times [0, d]) \cap \mathbf{R} \times \mathbf{R}_+^*) \ .$$

Lemma 1.7 is a consequence of the following simpler lemma, whose proof will be given below.

Lemma 5.1. *Let $\phi : [-d,d] \times [0,d]$ be a concave map. There exist $\epsilon \in (0,d]$ and $L > 0$ such that, for any $\epsilon' \leq \epsilon$, there exists a piecewise smooth curve γ of length at most L, which is an integral curve of U or V on each interval where it is smooth, and which goes from a point of* $\mathrm{Im}(\phi_{\epsilon'})$ *to* $\partial_R \phi_\epsilon$.

Proof of lemma 1.7. Suppose x is a concave point of $\partial_{I\!I\!I}\Sigma$. By definition, there exists a concave map $\phi : [-d,d] \times [0,d] \to \overline{\Sigma}$ at x.

Therefore, by lemma 5.1, there exists $\epsilon \in (0,d]$ such that, for each $\epsilon' \leq \epsilon$, $\mathrm{Im}(\phi_{\epsilon'})$ can be connected to $\partial_R \phi_\epsilon$ by a piecewise smooth curve γ of length at most L (for $I\!I\!I$), which is an integral curve of U or V on each smooth segment.

By remark 2.6, the length of γ for I is at most $L/\sqrt{K_2 - K_1}$, so $\mathrm{Im}(\phi_{\epsilon'})$ is at distance at most $L/\sqrt{K_2 - K_1}$ from $\partial_R \phi_\epsilon$ for I. This contradicts the fact that (Σ, I) is complete. □

Now for the proof of lemma 5.1. The proof is by contradiction, so we suppose that $\partial_{I\!I\!I}\Sigma$ is concave at a point x_0, with a (k,C)-concave map ϕ at x_0, and such that there is no sequence of piecewise asymptotic curves of bounded lengths starting from $\partial_R \phi_\epsilon$ (for some fixed $\epsilon > 0$) and ending arbitrarily close to x_0. We first state a remark which will be used later on.

Proposition 5.2. *For $C_0, k_0 > 0$, there exists $C_1 > 0$ (depending on C_0, k_0, τ_0 and K_5) such that, if ϕ is a (k_0, C_0)-concave map, then the "vertical" curves $\phi(\{x\} \times (0,d))$ have geodesic curvature bounded by C_1.*

Proof. Let X, Y be the vector fields and v, l be the functions on $\mathrm{Im}(\phi)$ such that:

$$\partial_1 \phi = vX \ , \quad \partial_2 \phi = lY \ .$$

Call $k := I\!I\!I(\tilde{\nabla}_Y X, Y)$ and $\kappa := I\!I\!I(\tilde{\nabla}_X X, Y)$. Then:

$$X.k = I\!I\!I(\tilde{\nabla}_X \tilde{\nabla}_Y X, Y) = I\!I\!I(\tilde{\nabla}_Y \tilde{\nabla}_X X, Y) + I\!I\!I(R^{\tilde{\nabla}}_{X,Y} X, Y) + I\!I\!I(\tilde{\nabla}_{[X,Y]} X, Y) \ .$$

But $[vX, lY] = 0$, so that:

$$[X,Y] + \frac{dl(X)}{l} Y - \frac{dv(Y)}{v} X = 0 \ ,$$

and therefore:

$$X.k = Y.I\!I\!I(\tilde{\nabla}_X X, Y) + \tilde{K} + \frac{dv(Y)}{v} I\!I\!I(\tilde{\nabla}_X X, Y) - \frac{dl(X)}{l} I\!I\!I(\tilde{\nabla}_Y X, Y) \ .$$

Now, by definition of τ:

$$\tilde{\nabla}_{vX}(lY) - \tilde{\nabla}_{lY}(vX) - [vX, lY] = \tau(vX, lY) \ ,$$

so that:

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X + \frac{dl(X)}{l} Y - \frac{dv(Y)}{v} X = \tau \ ,$$

and therefore:

$$\frac{dl(X)}{l} = k + I\!I\!I(\tau, Y) \ , \quad \frac{dv(Y)}{v} = -\kappa - I\!I\!I(\tau, X) \ .$$

Using this and (5) shows that:

$$X.k = Y.\kappa + \tilde{K} - \kappa(\kappa + I\!I\!I(\tau, X)) - k(k + I\!I\!I(\tau, Y)) \ .$$

Now the definition of a convex map and the bounds on $\tilde{K}$ and τ show that:

$$|X.k| \leq C_0 + K_5 + C_0 k_0 (C_0 k_0 + \tau_0) + |k|(|k| + \tau_0) \ .$$

This means that, if k is large at a point m, then it remains large in a neighborhood of m in the integral curve of X through m. Equation (5) shows that l would then vary a lot on this curve, and this would contradict the definition of a convex map (because $1 \leq \|\partial_2 \phi\| \leq C_0$). □

The next point in the proof of lemma 5.1 is to prohibit the existence of asymptotic curves with δ small, using lemma 4.3 and the following result:

Proposition 5.3. *For each $C > 0$, there exist $\epsilon(C) > 0$ such that, for any $\epsilon \leq \epsilon(C)$, if $\mathrm{diam}(\phi) \geq \epsilon$ and if $c : [-4\epsilon, 4\epsilon] \rightarrow \Sigma$ is a $Cy(c(0))$-quasi-geodesic, then c meets $\partial_R \phi_\epsilon$.*

Proof. Let $c : [0, L) \rightarrow \mathrm{Im}(\phi)$ be an ϵ-quasi-geodesic, with either $L = \infty$ or $L \in \mathbf{R}_+^*$ and $\lim_L c \in \partial\phi$. Note W the vector field on $\mathrm{Im}(\phi)$ defined by:

$$W := \frac{\partial_1 \phi}{\|\partial_1 \phi\|} \ .$$

Let $\alpha(t)$ be the angle between W and $c'(t)$, and $\alpha_0(t)$ the angle between W and the parallel transport at $c(t)$ of $c'(0)$ along $c([0, t])$.

By proposition 5.2:

$$\|\tilde{\nabla}_{\partial_2 \phi} W\| \leq C_1 \|\partial_2 \phi\| \ ,$$

while the definition of a (k, C)-concave map indicates that:

$$k \leq \langle \tilde{\nabla}_W W, JW \rangle \leq Ck \ .$$

As a consequence:

$$k \cos\alpha(t) - C_1 |\sin\alpha(t)| \leq \langle \tilde{\nabla}_{c'(t)} W, JW \rangle \leq Ck \cos\alpha(t) + C_1 |\sin\alpha(t)| \ ,$$

so that, by definition of a quasi-geodesic, if $\epsilon_0 := Cy(c(0))$, then:

$$k \cos\alpha_0(t) - C_1 |\sin\alpha_0(t)| - (Ck + C_1)\epsilon_0 \leq$$
$$\leq \alpha_0'(t) \leq Ck \cos\alpha_0(t) + C_1 |\sin\alpha_0(t)| + (Ck + C_1)\epsilon_0 \ .$$

We now suppose (without loss of generality) that $\alpha_0(0) \in [0, \pi/2]$. Let $\alpha_1 > 0$ be the smallest positive number such that:

$$k \cos(\alpha_1) - C_1 \sin(\alpha_1) - (Ck + C_1)\epsilon_0 \geq 0 \ .$$

α_1 exists if ϵ_0 is small enough (which happens if ϵ is small enough). Equation (5) indicates that, if $\alpha_0 \in [\alpha_1, \pi - \alpha_1]$ at a time t, then it remains there until the time L where c leaves $\mathrm{Im}(\phi)$; moreover, α_0 reaches $[\alpha_1, \pi - \alpha_1]$ before a fixed time t_0 (depending on C, K, etc) and, in the interval $[0, t_0]$, it remains above $-c_0\epsilon_0$, where $c_0 > 0$ is a constant depending also on C and k.

Now it is easy to check that:

$$\frac{\cos\alpha(t)}{C} \leq x'(t) \leq \cos\alpha(t) \ ,$$
$$\frac{\sin\alpha(t)}{C} \leq y'(t) \leq \sin\alpha(t) \ ,$$

so that:

$$\frac{\cos\alpha_0(t)}{C} - \epsilon_0 \leq x'(t) \leq \cos\alpha_0(t) + \epsilon_0 \ ,$$
$$\frac{\sin\alpha_0(t)}{C} - \epsilon_0 \leq y'(t) \leq \sin\alpha_0(t) + \epsilon_0 \ .$$

As a consequence of the lower bounds on α_0 and on y':

$$\forall t \in [0, L], \ y(t) \geq y(0) - (c_0 + 1)t_0\epsilon_0 \ ,$$

so that c must intersect $\partial_R \phi_\epsilon$ if ϵ is small enough. □

The same ideas also lead to the following statement, where we suppose that $c'(0)$ is not too horizontal, instead of supposing that $y(c(0))$ is not too small. The proof is left to the reader.

Proposition 5.4. *There exists $C_\angle > 0$ and $\epsilon_\angle > 0$ such that, for any $\epsilon \leq \epsilon_\angle$, if $\mathrm{diam}(\phi) \geq \epsilon$ and if $c : [-4\epsilon, 4\epsilon] \rightarrow \Sigma$ is a α_0-quasi-geodesic with $C_\angle \alpha_0 \leq \angle(\partial_1 \phi, c'(0)) \leq \pi/2$, then c meets $\partial_R \phi_\epsilon$.*

The situation is simpler if c is a geodesic instead of a quasi-geodesic:

Proposition 5.5. *There exist $C_3, \epsilon_3 > 0$ as follows. Let $\epsilon \leq \epsilon_3$, and suppose that $\mathrm{diam}(\phi) \geq \epsilon$. Let $g : [a, b] \to \mathrm{Im}(\phi_{2\epsilon})$ be a maximal geodesic segment in $\mathrm{Im}(\phi_{2\epsilon})$, with $a < 0 < b$ and $g(0) \in \mathrm{Im}(\phi_\epsilon)$. Then either $g(a)$ or $g(b)$ is on $\partial_R \phi_{2\epsilon}$, and $L(g) \in [\epsilon/C_3, C_3\epsilon]$. If $g'(0)$ is parallel to $\partial_1 \phi$, then both $g(a)$ and $g(b)$ are on $\partial_R \phi_{2\epsilon}$.*

Proof. The proof is similar to that of proposition 5.3; we call $\alpha_0(t) := \angle(W, g'(t))$, and we suppose that $\alpha_0(0) \in [0, \pi/2]$. Then:

$$k \cos \alpha_0(t) - C_1 |\sin \alpha_0(t)| \leq \alpha_0'(t) \leq Ck \cos \alpha_0(t) + C_1 |\sin \alpha_0(t)| \ .$$

It is the clear that α_0 will soon become positive; moreover, if we let $x(t) := x(g(t))$ and $y(t) := y(g(t))$, then:

$$\frac{\cos \alpha_0(t)}{C} \leq x'(t) \leq \cos \alpha_0(t) \ ,$$

$$\frac{\sin \alpha_0(t)}{C} \leq y'(t) \leq \sin \alpha_0(t) \ .$$

The second equation indicates that $y(t)$ remains positive while $g(t) \in \mathrm{Im}(\phi_\epsilon)$, and both equations taken together again show that g intersects $\partial_R \phi_\epsilon$ after time at most $C_3'\epsilon$ for some $C_3' > 0$.

The same equations apply for the segment of g where $t \leq 0$; after a bounded time, either g will have intersected $\partial \phi_{2\epsilon} \setminus \partial_R \phi_{2\epsilon}$, or y' will vanish. The same argument as above then shows that, in both cases, $t \mapsto g(-t)$ will meet $\partial \phi_{2\epsilon}$ after a time at most $C_3''\epsilon$ for some $C_3'' > 0$. This proves the upper bound on the length of g. The corresponding lower bound comes from the distance between $\mathrm{Im}(\phi_\epsilon)$ and the part of $\partial_R \phi_{2\epsilon}$ that g can intersect for $t > 0$.

Finally, the case where $g'(0)$ is parallel to $\partial_1 \phi$ is obtained by applying twice the argument for $t > 0$, which can be used in this case also for $t < 0$ because $-g'(0)$ is also directed towards the increasing values of y. □

From now on, we consider an integral curve $g : I \to \Sigma$ of U, where I is an interval, either of the form $[0, t_M]$ or $\mathbf{R}_+$. ϵ is a fixed positive number, on which more details are given below.

Definition 5.6. *Let $t \in I$; call γ_t the maximal geodesic segment directed by $V(g(t))$. Call E_g the subset of I containing all t such that γ_t intersects $\partial_R \phi_\epsilon$ on both sides at finite distance. For $t \in E_g$, call Ω_t the connected component of $\mathrm{Im}(\phi_\epsilon) \setminus \gamma_t$ which does not contain x_0 in its boundary.*

Proposition 5.7. *If $t \in E_g$ and $g'(t)$ is towards the interior of Ω_t, then, for all $t' \in I$ with $t' \geq t$, $t' \in E_g$, and $\Omega_{t'} \subset \Omega_t$.*

Proof. Note that, if ϵ is small enough, then, for any $t' \in E$, if $t'' > t'$ is close enough to t', then $\gamma_{t'} \cap \gamma_{t''} = \emptyset$. This comes from (3) and from corollary 3.2. This immediately implies that (Ω_t) is a decreasing family of subsets of $\mathrm{Im}(\phi_\epsilon)$. □

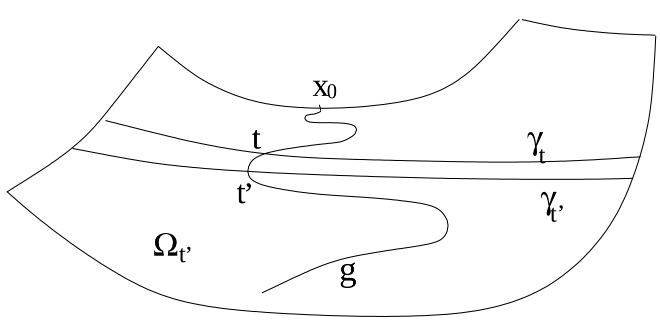

Figure 5.1

This is now used to prove that there exists an asymptotic curve going from x_0 to $\partial_R \phi_\epsilon$.

Proposition 5.8. *If ϵ is small enough, there exists an integral curve $g : \mathbf{R}_+ \to \mathrm{Im}(\phi_\epsilon)$ of U (or of V) such that $g(0) \in \partial_R\phi_\epsilon$ and that $\lim_{t\to\infty} g(t) = x_0$.*

Proof. Fix $\epsilon' > 0$. For $z \in (0, \epsilon')$, let c_z be the maximal integral curve of V in $\mathrm{Im}(\phi_{\epsilon'})$ containing $\phi(0, z)$. We consider two cases.

1. There exists $\epsilon > 0$ and a sequence $z_n \to 0$ such that, for each n, c_{z_n} has one end on $\partial_R\phi_\epsilon$.

If there exists n such that $\overline{c_{z_n}} \ni x_0$, then the proposition is proved. Otherwise, call D_n the connected component of $\mathrm{Im}(\phi_{\epsilon'})$ which does not contain x_0 in its closure. Let $D := \cup_n D_n$. Since the c_{z_n} are integral curves of V, they are pairwise disjoint (except when they coincide), so (maybe after taking a subsequence of (z_n)) (D_n) is an increasing sequence. Since $c_{z_n} \ni \phi(0, z_n) \to x_0$, it is then not difficult to prove that ∂D contains an integral curve of V connecting $\partial_R\phi_{\epsilon'}$ to x_0.

2. For all $\alpha > 0$, there exists $z_\alpha > 0$ such that, for $z \le z_\alpha$, c_z remains in $\mathrm{Im}(\phi_\alpha)$ and has both ends on $\partial\phi_\alpha \setminus \partial_R\phi_\alpha$.

Call m_z a point of c_z where y is maximal. Let $g_z : [0, L_z] \to \mathrm{Im}(\phi_{\epsilon'})$ be the maximal integral curve of U (or $-U$) with $g_z(0) = m_z$ and $g'(0)$ directed towards the increasing values of y. By definition of m_z, $V(g(0))$ is parallel to $\partial_1\phi$. Therefore, by proposition 5.5, the geodesic directed by $V(g(0))$ meets $\partial_R\phi_{\epsilon'}$ on both sides. With the notations above, this indicates that $0 \in E_{g_z}$, so that, by proposition 5.7, $E_{g_z} = [0, L_z]$. Thus $g_z(L_z) \in \partial_R\phi_{\epsilon'}$.

The proof then proceeds as in case (1.) above, because the (g_z) are disjoint and, after taking a sequence $(z_n) \to 0$, they converge to an integral curve of U connecting $\partial_R\phi_{\epsilon'}$ to x_0. □

Moreover, the rate of decrease of the area of Ω_t is bounded by $\sin\theta(g(t))$:

Proposition 5.9. *There exists $\lambda_4 > 0$ such that, if $t \in E$ is such that $g(t) \in \mathrm{Im}(\phi_{\lambda_4\epsilon})$, then:*

$$\left|\frac{\partial \mathrm{area}(\Omega_t)}{\partial t}\right| \ge \lambda_4 \sin\theta(g(t))\epsilon \ ,$$

and:

$$\liminf_{t'\to t^+} \frac{d(\gamma_{t'}, \gamma_t)}{t' - t} \ge \lambda_4 \sin\theta(g(t)) \ .$$

Proof. This is again a consequence of corollary 3.2, along with (3), which bounds the rate of variation of the direction of V along γ. □

For each $t \in \mathbf{R}_+$, we let $h_t : [-T_0, T_0] \to \Sigma$ be the integral curve of V with $h_t(0) = g(t)$. A direct consequence of the previous proposition is that the integral of $\sin\theta$ on g is finite, and this will be used now to show that many h_t are quasi-geodesics.

Proposition 5.10. *If ϵ is small enough, there exists $C > 0$ such that, for each $t_0 \ge C$, there exists $t \ge t_0$ such that h_t intersects $\partial_R\phi_\epsilon$.*

Proof. Since the integral of $\sin\theta$ on g is finite, equation (3) shows that:

$$\lim_{t,t'\to\infty} \angle(V(g(t')), \Pi(g_{|[t,t']}; V(g(t)))) = 0 \ .$$

One can therefore define a parallel vector field on V_0 as:

$$V_0(g(t)) := \lim_{t'\to\infty} \Pi(g_{|[t,t']}; V(g(t'))) \ ,$$

and, by (3):

$$|\angle(V_0(g(t)), V(g(t)))| \le \tau_1 \int_t^\infty \sin\theta(g(s))da \overset{t\to\infty}{\longrightarrow} 0 \ .$$

The same works for W because $\lim_\infty g = x_0$; set:

$$W_0(g(t)) := \lim_{t\to\infty} \Pi(g_{|[t,t']}; W(g(t'))) \ ,$$

and then:

$$\lim_{t\to\infty} \angle(W_0(g(t)), W(g(t))) = 0 \ .$$

Let $\alpha_0 := \angle(W_0, V_0)$; we suppose (without loss of generality) that $\alpha_0 \in [0, \pi/2]$. The proof will proceed differently according to whether $\alpha_0 = 0$ or $\alpha_0 > 0$.

If $\alpha_0 > 0$, remark that, since $\lim \infty g = x_0$, an elementary argument (as e.g. in the proof of proposition 5.3) shows that:

$$\lim_{t\to\infty} \int_t^{t+T_0} \cos\theta(g(s))ds = 0 \ .$$

Thus, for any fixed $t_0 \in \mathbf{R}_+$ and $\lambda > 0$, there exist $u, v \in \mathbf{R}_+$ such that:

- $t_0 \leq u \leq v \leq u + T_0$;
- $\theta(g(u)) \leq \lambda$ and $\theta(g(v)) \geq \pi - \lambda$;
- $\int_u^v \sin\theta(g(s))ds \leq \lambda$;
- $\angle(\partial_1\phi, V) \geq \alpha_0/2$ at $g(u)$.

Then:

$$\delta_{g_{|[u,v]}} \leq 2\lambda \ , \quad \sigma_{g_{[u,v]}} \leq \lambda \ .$$

According to lemma 4.3, h_u is a quasi-geodesic; proposition 5.4 then indicates that, if λ and ϵ are small enough, h_u intersects $\partial_R\phi_\epsilon$.

Consider now the case where $\alpha_0 = 0$. By proposition 5.9, there exists $c > 0$ so that, for t large enough:

$$y(g(t)) \geq c \int_t^\infty \sin\theta(g(s))ds \ .$$

But, again:

$$\lim_{t\to\infty} \int_t^{t+T_0} \cos\theta(g(s))ds = 0 \ ,$$

and the same argument as in the case $\alpha_0 = 0$ leads to the conclusion, but with proposition 5.4 replaced by proposition 5.3 to shows that h_u, which is a quasi-geodesic, actually intersects $\partial_R\phi_\epsilon$ if ϵ is small enough. □

The proof of lemma 5.1 obviously follows, because the conclusion of the previous proposition contradicts the hypothesis, made above, that the conclusion of lemma 5.1 does not hold.

6 The boundary is convex

This section contains the proof that Σ, with $I\!I\!I$ and $\tilde{\nabla}$, is convex in the sense of definition 1.3.

Lemma 1.8. *If Σ has no concave point, then it is convex.*

We need to make normal deformations of curves, while controling their curvature. Some of the tools needed here will be used again in the next section, to prove that convex surfaces have bounded area.

From now on, whenever we consider a smooth, convex curve g, we suppose that it is parametrized in such a way that $J_{I\!I\!I}g'$ is oriented towards the convex side of g.

Proposition 6.1. *Let $g : [0, L] \to \Sigma$ be a smooth curve with $\|g'\| \equiv 1$. Let $l : [0, L] \to \mathbf{R}$ be a smooth function. The first order variation of the curvature κ of g in a normal deformation of g which is defined at $g(s)$ by the vector $l(s)Jg'(s)$ is:*

$$\dot{\kappa} = l(\tilde{K} + \kappa(\kappa + \tau(g'(s)))) + X.X.l - X.(l\tau(Jg'(s))) \ .$$

Proof. By linearity, it is enough to prove this proposition when l is positive. Let $(g_s)_{s\in[0,1]}$ be a one parameter family of curves such that $g_0 = g$ and that:

$$\partial g_s(t)/\partial s \equiv l J g'(s)/\|g'\| \ .$$

To simplify somewhat the notations, we call v_s the speed of g_s, that is:

$$v_s(t) := \left\| \frac{\partial g_s(t)}{\partial t} \right\|_{I\!I\!I} \ .$$

We also call X the unit vector along $g'_s(t) := \partial g_s(t)/\partial t$, and $Y := JX$. Therefore:

$$\partial_s(g_s(t)) = l_s Y$$

for some function $l_s(t)$ such that $l_0 = l$. Moreover, $\kappa = \kappa_s(t) = I\!I\!I(\tilde{\nabla}_X X, Y)$.

By definition of the torsion τ of $\tilde{\nabla}$,

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \tau \ ,$$

while, since (s,t) define a coordinate system on a domain of Σ:

$$[vX, lY] = 0 \ .$$

Set $\lambda := \ln(l)$ and $\nu := \ln(v)$, the previous equation becomes:

$$[X, Y] + (X.\lambda)Y - (Y.\nu)X = 0 \ .$$

Let $\tau_X := \langle \tau, X\rangle$ and $\tau_Y := \langle \tau, Y\rangle$. Then:

$$\begin{aligned}
Y.\kappa &= I\!I\!I(\tilde{\nabla}_Y \tilde{\nabla}_X X, Y) \\
&= \tilde{K} + I\!I\!I(\tilde{\nabla}_X \tilde{\nabla}_Y X - \tilde{\nabla}_{[X,Y]} X, Y) \\
&= \tilde{K} + I\!I\!I(\tilde{\nabla}_X (\tilde{\nabla}_X Y - [X, Y] - \tau) + (X.\lambda)\tilde{\nabla}_Y X - (Y.\nu)\tilde{\nabla}_X X, Y) \\
&= \tilde{K} + I\!I\!I(\tilde{\nabla}_X(-\kappa X + (X.\lambda)Y - (Y.\nu)X - \tau_X X - \tau_Y Y), Y) + \\
&\quad +(X.\lambda) I\!I\!I(\tilde{\nabla}_X Y - [X, Y] - \tau, Y) - \kappa(Y.\nu) \\
&= \tilde{K} - \kappa^2 + X.X.\lambda - \kappa(Y.\nu) - \kappa\tau_X - X.\tau_Y + (X.\lambda)^2 - (X.\lambda)\tau_Y - \kappa(Y.\nu) \ ,
\end{aligned}$$

and, since $Y.\nu = -\kappa - \tau_X$ by (5) and (5):

$$\begin{aligned}
Y.\kappa &= \tilde{K} + X.X.\lambda + \kappa^2 + \kappa\tau_X - X.\tau_Y + (X.\lambda)^2 - (X.\lambda)\tau_Y \\
&= \tilde{K} + X.X.\lambda + (X.\lambda)^2 + \kappa(\kappa + \tau_X) - (X.\tau_Y) - \tau_Y(X.\lambda) \ ,
\end{aligned}$$

so that:

$$\begin{aligned}
\partial_s \kappa &= l(Y.\kappa) \\
&= l(\tilde{K} + \kappa(\kappa + \tau_X)) + X.X.l - X.(l\tau_Y) \ ,
\end{aligned}$$

which is the formula we need. □

As a consequence, we find an inequality:

Corollary 6.2. *The rate of variation of κ is bounded from below by:*

$$\dot{\kappa} \geq \frac{l}{4}((4K_4 - \tau_0^2) + \tau(Jg')^2) + l'' - (l\tau(Jg'))' \ .$$

If the curvature of g is bounded from above by κ_M, then $\dot{\kappa}$ is also bounded from above:

$$\left[\frac{l}{4}((4K_4 - \tau_0^2) + \tau(Jg')^2) + l'' - (l\tau(Jg'))'\right] + l\left((K_5 - K_4) + \kappa_M^2 + \kappa_M\tau_0 + \tau_0^2\right) \geq \dot{\kappa} \ .$$

Proof. For any $\kappa \in [0, \kappa_M]$, we have:

$$\kappa_M(\kappa_M + \tau_0) \geq \kappa(\kappa + \tau_X) \geq \frac{\tau_X^2}{4} \ .$$

Moreover, $\tau_X^2 + \tau_Y^2 = |\tau|^2 \leq \tau_0^2$ and $K_4 \leq \tilde{K} \leq K_5$, so that:

$$K_5 + \kappa_M(\kappa_M + \tau_0) \geq \tilde{K} + \kappa(\kappa + \tau_X) \geq K_4 - \frac{\tau_0^2}{4} + \frac{\tau_Y^2}{4} \ ,$$

and the corollary follows. □

This means that trying to deform curves leads to a natural question on solutions of differential equations; we will need the following proposition.

Proposition 6.3. *For each $\epsilon > 0$ small enough, there exists $M_0 \geq 1$ and $S_1 \geq S_0 > 0$ such that, if $u \in C^\infty(\mathbf{R}, [-1/\epsilon, 1/\epsilon])$, there exists $s_0, s_1 \in [S_0, S_1]$ and $y \in C^\infty([0, s_1], [0, M_0])$ such that:*

$$y'' = (yu)' - (\epsilon + \frac{u^2}{4})y \ ,$$

with:

$$\begin{gathered} y(s) \in [1, M_0] \text{ for } s \in [0, s_0] \ , \\ y(0) = 1, \ y'(0) = u(0) + 4 \ , \\ y(s_0) = 1, \ y'(s_0) \leq u(s_0) + 4 \ , \\ y(s_1) = 0, \ y'(s_1) \leq 0 \ . \end{gathered}$$

Moreover, for each $s \in [0, s_0]$, $|y'(s)| \leq M_0$.

Proof. Let $z = y' - yu$. The relation (5) becomes:

$$\left\{ \begin{array}{rcl} y' & = & yu + z \\ z' & = & -(\epsilon + u^2/4)y \ . \end{array} \right.$$

Let:

$$X := \begin{pmatrix} y \\ z \end{pmatrix} \text{ and } m := \begin{pmatrix} u & 1 \\ -(\epsilon + u^2/4) & 0 \end{pmatrix} \ .$$

The relation now is:

$$X'(s) = m(s)X(s) \ ,$$

so, for $s \geq 0$:

$$X(s) = \exp(sM(s))X(0) = \exp\left(s \begin{pmatrix} F(s) & 1 \\ -(\epsilon + \tilde{F}^2/4) & 0 \end{pmatrix}\right) X(0) \ ,$$

with:

$$F(s) = \frac{1}{s}\int_0^s u(r)dr \ , \ \tilde{F}(s) = \sqrt{\frac{1}{s}\int_0^s u(r)^2 dr} \ .$$

Now the eigenvalues of $M(s)$ are the roots of:

$$X(X - F) + \left(\epsilon + \frac{\tilde{F}(s)^2}{4}\right) = 0 \ .$$

From the Cauchy-Schwarz theorem, $\tilde{F}(s)^2 \geq F(s)^2$, so $4\epsilon + \tilde{F}^2 - F^2 \geq 4\epsilon > 0$, and the eigenvalues are $\mu_\pm = \alpha \pm i\beta$, where:

$$|\alpha| = \frac{|F|}{2} \leq \frac{1}{2\epsilon} \ , \quad |\beta| = \sqrt{\epsilon + \frac{\tilde{F}^2 - F^2}{4}} \geq \sqrt{\epsilon} \ .$$

The associated eigenvectors are $v_\pm = (1, -\alpha \pm i\beta)$.

Suppose now that $X(0) = (1,4)$. Then $y'(0) > 0$, and $y(s)$ remains above 1 until after time $s_0 > 0$. But $\beta \geq \sqrt{\epsilon}$, so it is clear that, after a time $s_1 \leq \pi/\sqrt{\epsilon}$, $y(s)$ will become negative, and this provides an upper bound for s_0 and for s_1:

$$s_0 \leq s_1 \leq S_1 := \frac{\pi}{\sqrt{\epsilon}}$$

The decomposition of $(1,4)$ on the basis (v_+, v_-) is:

$$(1,4) = \alpha_+ v_+ + \alpha_- v_- = \left(\frac{1}{2} - i\frac{\alpha+4}{2\beta}\right) v_+ + \left(\frac{1}{2} + i\frac{\alpha+4}{2\beta}\right) v_- \ ,$$

and this gives an upper bound for $\alpha_\pm$:

$$|\alpha_\pm|^2 \leq \frac{1}{4} + \frac{(|\alpha|+4)^2}{4\beta^2} \leq \frac{1}{4} + \frac{1}{4\epsilon}\left(\frac{2}{\epsilon} + 4\right)^2 \ .$$

Moreover, β also has an upper bound because $|u| \leq 1/\epsilon$, and this gives a lower bound S_0 for s_0.

Using the upper bound on s_0 (equation (5)) and on α ($|\alpha| \leq 1/2\epsilon$), we see that, for any $s \in [0, t_0]$:

$$\begin{aligned} y(s) &\leq |X(s)| \\ &\leq (|\alpha_+| + |\alpha_-|)\exp(s_0) \\ &\leq \sqrt{1 + \frac{1}{\epsilon}\left(\frac{2}{\epsilon}+4\right)^2}\exp\left(\frac{\pi}{\sqrt{\epsilon}}\right) \ . \end{aligned}$$

In addition, $y(s) > 0$, so that $z'(s) < 0$ for $s \in [0, s_0]$, and it follows that $z(s_0) \leq z(0) = 4$. The function y therefore verifies the conclusions of proposition 6.3. $\square$

Corollary 6.4. *For any $\kappa_M > 0$, there exist $L_1 > 0, C_1 > 0$ and $M_1 \geq 1$ as follows. Let $\kappa_m \in [0, \kappa_M]$, and let $g_0 : [0, L] \to \Sigma$ be a convex curve parametrized at speed one, with curvature $\kappa \in [\kappa_m, \kappa_M]$, and with $L \in (0, L_1]$. Then there exists $T_1 > 0$ and a deformation $(g_t)_{t\in[0,T_1]}$ such that:*

1. *for each $t \in [0, T_1]$, g_t is a convex curve with curvature $\kappa \in [\kappa_m + t/C_1, \kappa_M + tC_1]$;*
2. *for any $t \in [0, T_1]$ and $s \in [0, L]$, $(\partial_t g_t)(s)$ is parallel to the unit normal N to g_t, and $\langle (\partial_t g_t)(s), N\rangle \in [1, M_1]$;*
3. *for each $t \in [0, T_1]$ and $s \in [0, L]$, $|\partial\kappa/\partial t| \leq C_1$.*

Proof. Let $\epsilon_0 := 4K_4 - \tau_0^2$, so that $\epsilon_0 > 0$ by lemma 1.1. By proposition 6.3, if L is small enough, there exists a function $l : [0, L] \to [1, M_0]$ such that, on $[0, L]$:

$$l'' + \left(\frac{\epsilon_0}{2} + \frac{\tau(Jg_0')^2}{4}\right) l - (l\tau(Jg_0'))' = 0 \ ,$$

so that:

$$l'' + \left(\epsilon_0 + \frac{\tau(Jg_0')^2}{4}\right) l - (l\tau(Jg_0'))' = \frac{\epsilon_0 l}{2} \in \left[\frac{\epsilon_0}{2}, \frac{\epsilon_0 M_0}{2}\right] \ .$$

For t small enough and $s \in [0, L]$, set:

$$g_t(s) = \exp_{g_0(s)} tN(s) \ ,$$

where $N(s)$ is the unit normal to g_0 at $g_0(s)$ towards the convex side of the complement. This defines, for t small enough, a smooth curve g_t.

Let $R_0 := (K_5 - K_4) + \kappa_M^2 + \kappa_M\tau_0 + \tau_0^2$. Corollary 6.2 shows that, for $t = 0$:

$$\dot{\kappa} \in \left[\frac{\epsilon_0}{2}, \frac{M_0(\epsilon_0 + R_0)}{2}\right] \ .$$

Then, by compactness, for t small enough, $\dot{\kappa} \in [\epsilon_0/4, M_0(\epsilon_0 + R_0)]$, and the corollary follows. $\square$

Corollary 6.5. *There exists $\epsilon_2 > 0$, $C_2 > 0$ and $L_2 > 0$ as follows. Suppose that $L \in (0, L_2]$, and let $g : [0, L] \to \Sigma$ be a geodesic segment. Let $d_0 := d_{I\!I\!I}(g([0, L]), \partial_{I\!I\!I}\Sigma)$, and suppose that $d_0 \leq \epsilon_2$. Suppose moreover that $d_{I\!I\!I}(g(0), \partial_{I\!I\!I}\Sigma) \geq C_2 d_0$ and that $d_{I\!I\!I}(g(L), \partial_{I\!I\!I}\Sigma) \geq C_2 d_0$. Then $(\Sigma, I\!I\!I, \tilde{\nabla})$ has a concave point.*

Proof. Apply corollary 6.4 recursively to obtain a (k, C)-concave map $\phi : [-d, d] \times [0, d] \to \overline{\Sigma}$ such that $\phi([-d, d] \times \{d\})$ is a segment of g. □

Proof of lemma 1.8. Suppose that $(\Sigma, I\!I\!I, \tilde{\nabla})$ is not convex. Then, by definition 1.3, there exists a sequence of geodesic segments $\gamma_n : [0, L] \to \Sigma$ such that $(\gamma_n(s))_{n\in\mathbf{N}}$ converges in $\overline{\Sigma}$ for each $s \in [0, L]$, with $\lim_{n\to\infty} \gamma_n(s') \in \partial_{I\!I\!I}\Sigma$ for some $s' \in (0, L)$ but $\lim_{n\to\infty} \gamma_n(0) \in \Sigma$. Set:

$$s_0 := \inf\{s \in [0, L] \mid \lim_{n\to\infty} \gamma_n(s) \in \partial_{I\!I\!I}\Sigma\} .$$

Remark that there exists $\epsilon_3 \in (0, \max(s_0/3, L_2/2))$such that, for each $n \in \mathbf{N}$, $B(\gamma_n(s_0), 3\epsilon_3)\setminus\gamma_n([0, L])$ has at least two connected components, one of which is a half-disk which does not meet $\partial_{I\!I\!I}\Sigma$. Otherwise, each ball centered at $\gamma_n(s_0)$ would meet $\partial_{I\!I\!I}\Sigma$ on each side of γ_n for each $n \in \mathbf{N}$, and then there could be no path joining $\gamma_n(0)$ to $\gamma_n(L)$ (for n large enough) in Σ, a contradiction. We suppose that the half-disk which does not meet $\partial_{I\!I\!I}\Sigma$ is always on the same side of γ_n as $J\gamma_n'(s_0)$.

By definition of s_0, $\gamma_n([0, s_0 - \epsilon_3])$ remains in a compact subset of Σ, so that there exists $\epsilon_4 > 0$ so that, for each $n \in \mathbf{N}$, $d_{I\!I\!I}(\gamma_n([0, s_0 - \epsilon_3]), \partial_{I\!I\!I}\Sigma) \geq 2\epsilon_4$. We call:

$$\Omega := \{x \in \Sigma \mid \exists n \in \mathbf{N}, d_{I\!I\!I}(x, \gamma_n([0, s_0 - \epsilon_3])) \leq \epsilon_4\} ,$$

so that $d_{I\!I\!I}(\Omega, \partial_{I\!I\!I}\Sigma) \geq \epsilon_4$.

Let $\theta \in (0, \pi)$. Call $s(\theta)$ the supremum of all $s \in [0, s_0]$ such that, for any $n \in \mathbf{N}$ and any $t \in [0, s]$, the maximal geodesic starting from $\gamma_n(s)$ with $\angle(\gamma_n'(s), g'(0)) = \theta - \pi$ does not reach $\partial_{I\!I\!I}\Sigma$ before time at least ϵ_3. Then, clearly:

$$\limsup_{\theta\to 0^+} s(\theta) = s_0 .$$

For $s \in [s_0 - \epsilon_3, s_0]$, $s < s(\theta)$, and $n \in \mathbf{N}$, the geodesic segment $\overline{g}$ starting from $\gamma_n(s)$ with $\angle(\gamma_n'(s), g'(0)) = \theta$ also does not reach $\partial_{I\!I\!I}\Sigma$ before time at least ϵ_3, because it remains in a half-disk bounded by γ_n and of radius $3\epsilon_3$. Let $g_{n,\theta,s} : [-\epsilon_3, \epsilon_3] \to \Sigma$ be the geodesic segment with $g_{n,\theta,s}(0) = \gamma_n(s)$ and $\angle(\gamma_n'(s), g_{n,\theta,s}'(0)) = \theta$.

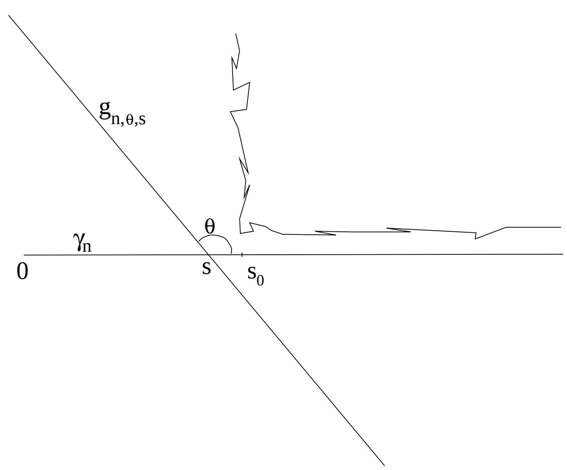

Figure 6.1

Now it is easy to check that, if θ is smaller than some fixed θ_4, then, for any $n \in \mathbf{N}$ and any $s \in [s_0 - \epsilon_3, s(\theta)]$, $d_{I\!I\!I}(g_{n,\theta,s}(-\epsilon_3), \gamma_n([0, s_0 - \epsilon_3])) \leq \epsilon_4$; then $g_{n,\theta,s}(-\epsilon_3) \in \Omega$, so that $d_{I\!I\!I}(g_{n,\theta,s}(-\epsilon_3), \partial_{I\!I\!I}\Sigma) \geq \epsilon_4$.

On the other hand, one can check that $d_{I\!I\!I}(g_{n,\theta_4,s}(\epsilon_3), \gamma_n)$ is bounded below by some fixed $\epsilon_5 > 0$ depending only on θ_4 (and K_5, τ_0). Since $g_{n,\theta_4,s}([0, \epsilon_3])$ remains in a half-disk bounded by γ_n and of

radius $3\epsilon_3$, this shows that $d_{III}(g_{n,\theta_4,s}(\epsilon_3), \partial_{III}\Sigma) \geq \epsilon_5$. Finally, by definition of $s(\theta_4)$, we can choose n and s so that $d_{III}(g_{n,\theta_4,s}([-\epsilon_3, \epsilon_3]), \partial_{III}\Sigma) \leq \min(\epsilon_3, \epsilon_5)/C_2$.

We can therefore apply corollary 6.5 to finish the proof. □

7 The area is bounded

In this section, we assume that $(\Sigma, III, \tilde{\nabla})$ is convex (as in definition 1.3) and has curvature $\tilde{K} \geq K_4$ and torsion $\|\tilde{\tau}\| \leq \tau_0$, with $4K_4 > \tau_0^2$. We will prove lemma 1.9, which states that (Σ, III) has bounded area. This will be achieved through the following lemmas:

Lemma 7.1. *For any $\epsilon > 0$, there exists a simply connected domain $\Omega \subset \Sigma$, $\overline{\Omega} \subset \Sigma$, with locally convex boundary, such that $d_{III}(\partial_{III}\Omega, dr_{III}\Sigma) \leq \epsilon$.*

Lemma 7.2. *(Ω, III) can not be complete.*

Lemma 7.3. *$\partial_{III}\Omega$ can not have a non-compact component.*

Lemma 7.4. *If $\partial_{III}\Omega$ is a closed curve, then the area of Ω is at most $2\pi/K_4$.*

Lemma 7.4 is an immediate consequence of the Gauss-Bonnet theorem, so that the rest of this section contains the proofs of lemmas 7.1, 7.2 and 7.3. Lemma 1.9 follows: by lemma 7.1, any compact subset of Σ should be contained in a domain Ω with locally convex boundary, which should have area at most $2\pi/K_4$ by lemmas 7.2, 7.3 and 7.4.

Proof of lemma 7.1. Let $\mathcal{E}$ be the set of open simply connected domains $\Omega \subset \Sigma$ such that $\partial_{III}\Omega \setminus \partial_{III}\Sigma$ is locally convex, and that $d_{III}(\partial_{III}\Omega, \partial_{III}\Sigma) \leq \epsilon$. $\mathcal{E}$ is ordered by inclusion. Let Ω_0 be a minimal element of $\mathcal{E}$. We want to prove that $\overline{\Omega}_0 \subset \Sigma$; we proceed by contradiction, and suppose that there exists a point $x_0 \in \partial_{III}\Omega_0 \cap \partial_{III}\Sigma$.

Let $x_1 \in \Omega$ be such that $d(x_0, x_1) \leq \epsilon_0/2$ in Ω_0. Let $c : [0, L) \to \Omega_0$ be a smooth curve of length L with $c(0) = x_1$ and $\lim_{t\to L} c(t) = x_0$, with $L \leq t_g$, where t_g comes from corollaries 3.2 and 3.3. Let t_0 be the supremum of all $t \in [0, L)$ such that there exists a one-parameter family $(g_t)_{t\in[0,t]}$ of geodesic segments, with g_t going from $c(0)$ to $c(t)$. If $t_0 < L$, then:

$$\lim_{t\to t_0} d_{III}(g_t, \partial_{III}\Omega) = 0 \ .$$

If $t_0 < L$, then, as $t \to t_0$, g_t would approach either a point of $\partial_{III}\Omega_0 \setminus \partial_{III}\Sigma$ — but this is impossible because $\partial_{III}\Omega_0 \setminus \partial_{III}\Sigma$ is locally convex – or a point of $\partial_{III}\Omega_0 \cap \partial_{III}\Sigma$ – and this is impossible by definition 1.3. Therefore, $t_0 = L$, and there exists a geodesic segment $g := g_L : [0, l) \to \Omega_0$ with $g(0) = x_1$ and $\lim_{t\to l} g(t) = x_0$.

For $t \in [0, l)$ and $\theta \in [-\pi, \pi]$, let $\gamma_{t,\theta}$ be the maximal geodesic in Σ with $\gamma_{t,\theta}(0) = g(t)$ and $\angle(g'(t), \gamma'_{t,\theta}(0)) = \theta$. So $\gamma_{t,\theta}$ is a smooth map from $(-a_{t,\theta}, b_{t,\theta})$ to Σ, with $a_{t,\theta}, b_{t,\theta} \in \mathbf{R}_+^* \cup \{\infty\}$. Let:

$$E_t := \{\theta \in [-\pi, \pi] \mid a_{t,\theta} \leq b_{t,\theta}\},\ F_t := \{\theta \in [-\pi, \pi] \mid a_{t,\theta} \geq b_{t,\theta}\} \ .$$

For $t > l/2$, $0 \in F_t$, while $\pi \in E_t$. Moreover, both E_t and F_t are closed, so there exists $\theta_t \in E_t \cap F_t$, which means that $a_{t,\theta_t} = b_{t,\theta_t} \in \mathbf{R}_+ \cup \{\infty\}$.

First suppose that there exists a sequence $t_n \to l$ such that $a_{t_n,\theta_{t_n}}$ remains bounded. Then there exists t such that a_{t,θ_t} is finite and that the geodesic segments γ_{t,θ_t} remains within distance at most $\epsilon/2$ of $\partial_{III}\Sigma$ because $(\Sigma, III, \tilde{\nabla})$ is convex. $\Omega_0 \setminus \gamma_{t,\theta_t}$ has at least two connected components, one of which, Ω_1, is in $\mathcal{E}$: it is convex, and its boundary remains within distance at most ϵ of $\partial_{III}\Sigma$. But this contradicts the minimality of Ω_0, and this finishes the proof in this case.

Now suppose that $a_{t,\theta} \to \infty$. Since Σ has a convex boundary, for any $a > 0$ and any $\epsilon' > 0$, there exists t close to l such that $\gamma_{t,\theta_t}([-a, a])$ remains within distance ϵ' of $\partial_{III}\Sigma$. Call g_0 the restriction of γ_{t,θ_t} to $\mathbf{R}_+$, and apply proposition 3.6. It shows that, if a is large enough and ϵ' small enough, there exists

$\theta > 0$ such that g_θ (defined as in proposition 3.6) remains within distance ϵ of $\partial_{\mathit{III}}\Sigma$ (because it remains close to g_0, point (2) of 3.6) but goes from $\gamma_{t,\theta_t}(0)$ to $\partial_{\mathit{III}}\Sigma$ (because of point (3.) of 3.6).

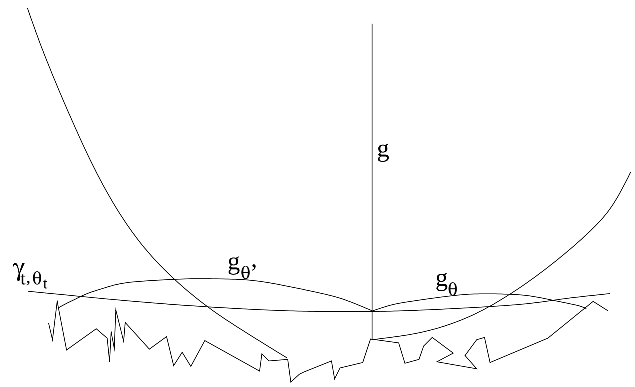

Figure 7.1

Then do the same for the geodesic ray g_0' defined by $g_0'(s) = \gamma_{t,\theta_t}(-s)$, to obtain another similar geodesic segment $g_{\theta'}'$ which remains within distance ϵ of $\partial_{\mathit{III}}\Sigma$, and goes from $\gamma_{t,\theta_t}(0)$ to $\partial_{\mathit{III}}\Sigma$.

The rest of the proof can be done as in the case where a_{t,θ_t} remains bounded, because $g_\theta \cup g_{\theta'}'$ "cuts" a convex domain in Ω whose boundary remains within distance ϵ of $\partial_{\mathit{III}}\Sigma$, a contradiction. $\Box$

We now come back to deformations of convex curves, as in the previous section. The following is a consequence of proposition 6.3.

Proposition 7.5. *For any $\epsilon > 0$ small enough, there exists M_1' as follows. Choose $N_1 > 0$. For any $u \in C^\infty(\mathbf{R}, [-1/\epsilon, 1/\epsilon])$, there exists $y \in C^0([0, s_0], [0, M_1'])$ which:*

1. *is piecewise C^∞;*
2. *vanishes outside $[-N_1 - M_1', N_1 + M_1']$;*
3. *is at least 1 in $[-N_1, N_1]$;*
4. *satisfies (as a distribution):*
$$y'' \geq (yu)' - (\epsilon + \frac{u^2}{4})y \ ;$$
5. *is M_1'-Lipschitz.*

Proof. Let $x_0 := -N_1$. First apply proposition 6.3 after translating the origin to $-N_1$ in $\mathbf{R}$, so as to obtain $x_1 \in [-N_1 + S_0, -N_1 + S_1]$ and a solution $y_1 : [x_0, x_1] \to [1, M_0]$ of (5) with:

$$y_1(x_0) = y_1(x_1) = 1 \ \text{ and } \ y_1'(x_0) = u(x_0) + 4, \ y_1'(x_1) \leq u(x_1) + 4 \ .$$

Apply proposition 6.3 again, now after a translation of the origin to x_1; this provides us with $x_2 \in [x_1 + S_0, x_1 + S_1]$ and with a solution $y_2 : [x_1, x_2] \to [1, M_0]$ of (5) with:

$$y_2(x_1) = y_2(x_2) = 1 \ \text{ and } \ y_2'(x_1) = u(x_1) + 4, \ y_2'(x_2) \leq u(x_2) + 4 \ .$$

Repeat this procedure to find a sequence $x_3 \leq \cdots \leq x_N$ with $x_{k+1} - x_k \in [S_0, S_1]$ and $x_{N-1} \leq N_1 < x_N$, and functions $y_k : [x_{k-1}, x_k] \to [1, M_0]$ which are solutions of (5) with:

$$y_k(x_{k-1}) = y_k(x_k) = 1 \ \text{ and } \ y_k'(x_{k-1}) = u(x_{k-1}) + 4, \ y_k'(x_k) \leq u(x_k) + 4 \ .$$

Apply proposition 6.3 once more to find $x_{N+1} \in [x_N + S_0, x_N + S_1]$ and a solution $y_{N+1} : [x_N, x_{N+1}] \to [0, M_0]$ of (5) with:

$$y_{N+1}(x_N) = 1 \ , \ \ y_{N+1}(x_{N+1}) = 0 \ , \ \ y_{N+1}'(x_N) = u(x_N) + 4 \ , \ \ y_{N+1}'(x_{N+1}) \leq 0 \ .$$

Do the same to find $x_{-1} \in [x_0 - S_1, x_0 - S_0]$ and a solution $y_{-1} : [x_N, x_{N+1}] \to [0, M_0]$ of (5) with:

$$y_{-1}(x_0) = 1 \ , \quad y_{-1}(x_{-1}) = 0 \ , \quad y'_{-1}(x_0) = 0 \ , \quad y'_{-1}(x_{-1}) \geq 0 \ .$$

Now define $y : \mathbf{R} \to \mathbf{R}_+$ as the function whose restriction to $[x_k, x_{k+1}]$ is y_{k+1} for $-1 \leq k \leq N$, and which is zero outside $[x_{-1}, x_{N+1}]$. Note that, if $M'_1 := 2S_1$, then y vanishes outside $[-N_1 - M'_1, N_1 + M'_1]$. Moreover, it is clear that y is a (weak) solution of (5). $\square$

The previous proposition provides the tool needed to deform convex curves while increasing their curvature.

Corollary 7.6. *There exist $c_1 > 0$ as follows. Let $\kappa_m \in \mathbf{R}_+$, and let $g_0 : \mathbf{R} \to \Sigma$ be a smooth, convex curve parametrized at speed one, with curvature $\kappa \geq \kappa_m$ as a measure. Let $N_1 > 0$. Then there exists $T > 0$ and a deformation $(g_t)_{t \in [0,T]}$ such that, for each $t \in [0, T]$:*

1. *g_t is a convex curve with curvature $\kappa \geq \kappa_m$, and $\kappa \geq \kappa_m + tc_1$ along $g_t([-N_1, N_1])$;*
2. *$g_t \equiv g_0$ outside $[-N_1 - M'_1, N_1 + M'_1]$;*
3. *for each $s \in \mathbf{R}$, either $(\partial_t g_t)(s)$ is zero, or its orthogonal is a support direction of g_t, and its norm is at most M'_1.*

Proof. Let $(g_n)_{n \in \mathbf{N}^*}$ be a sequence of smooth curves, $g_n : \mathbf{R} \to \Sigma$, such that:

- $\forall s \in \mathbf{R}, \lim_{n \to \infty} g_n(s) = g_0(s)$;
- for $n \leq m$, g_n lies entirely on the concave side of g_m;
- the curvature of g_n is at least $-\alpha_n < 0$, where $\lim_{n \to \infty} \alpha_n = 0$.

The existence of such an approximating sequence is not too difficult to prove. The (g_n) are not parametrized at speed one.

We suppose (without loss of generality) that, for $n \in \mathbf{N}^*$ and $s \in \mathbf{R}$, $Jg'_n(s)$ is towards g_0. For $n \in \mathbf{N}^*$ and $s \in \mathbf{R}$, let:

$$u_n(s) := \tau(Jg'_n(s)) \ .$$

Apply corollary 7.5 to obtain a sequence of piecewise smooth, M'_1-Lipschitz functions $(y_n)_{n \in \mathbf{N}^*}$ with:

$$y''_n \geq (y_n u_n)' - \left(\epsilon + \frac{u_n^2}{4}\right) y_n \ ,$$

with $y_n(s) = 0$ when $s \notin [-N_1 - M'_1, N_1 + M'_1]$ and $y_n(s) \in [1, M'_1]$ when $s \in [-N_1, N_1]$. Since the y_n are Lipschitz, we can (by taking a subsequence) suppose that they are C^0-converging to a Lipschitz function $y : \mathbf{R} \to [0, M'_1]$.

Let $T' \in \mathbf{R}_+ \cup \{\infty\}$ be the largest t such that, for each $n \in \mathbf{N}^*$ and each $s \in [-N_1, N_1]$, $\exp^{\tilde{\nabla}}_{g_n(s)}(tJg'_n(s))$ is defined. Then $T' > 0$ by compactness. For $n \in \mathbf{N}^*$ and $s \in [0, T')$, let:

$$h_{n,t}(s) := \exp^{\tilde{\nabla}}_{g_n(s)}(tJg'_n(s)) \ .$$

For $n \in \mathbf{N}^*$ and $t \in [0, T')$, $h_{n,t}$ is a curve which might not be embedded, but which, for t small enough, is immersed. It differs from g_n only in $[-N_1 - M'_1, N_1 + M'_1]$. Moreover, corollary 6.2 and a simple compactness argument show that there exist $N \in \mathbf{N}^*$, $T \in (0, T')$ and $c > 0$ such that the curvatures $\kappa_{n,t}$ of the curves $h_{n,t}$ satisfy:

$$\forall n \geq N, \forall t \in [0, T), \forall s \in \mathbf{R}, \kappa_{n,t}(s) \geq \kappa_m + ct - \epsilon_n \ ,$$

where the left-hand side is a measure and $\epsilon_n \to 0$.

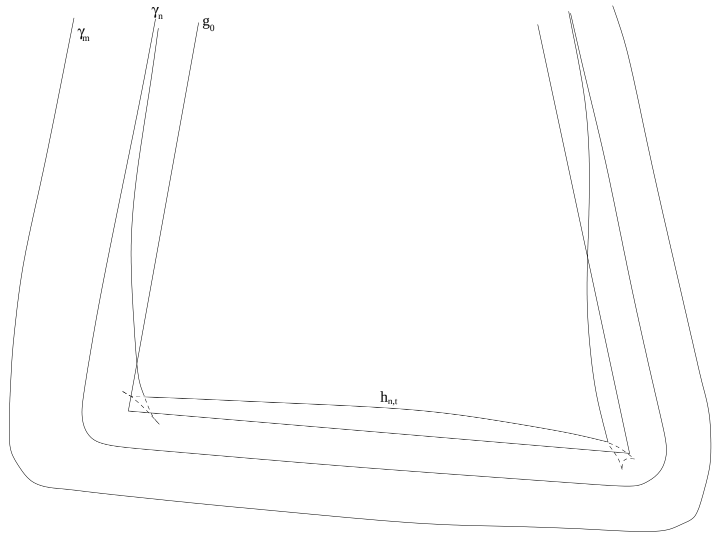

Figure 7.2

Since the $h_{n,t}$ are curves and differ from g_n only in a compact set, they separate Σ into several connected components, two of which are non compact; we call $\Omega_{n,t}$ the non-compact connected component of $\Sigma \setminus h_{m,t}$ whose intersection with the concave side of g_n is (empty or) compact. Equation (5) shows that the boundary of $\Omega_{n,t}$ in Σ is locally convex, with curvature at least $\kappa_m + ct - \epsilon_n$.

Finally, for $t \in [0, T)$, set:

$$g_t := \partial \left(\cap_{n=0}^{\infty} \Omega_{n,t}\right) .$$

It is not difficult to check that (g_t), with an adequate parametrization, satisfies the conclusion of corollary 7.6. □

As a consequence, the same kind of deformation can be done not only for small t, but for all t:

Corollary 7.7. *Let Ω be a closed subset of Σ with locally convex boundary. Suppose that some connected component of $\partial\Omega$ is a complete, non compact curve, parametrized at speed one by $c_0 : \mathbf{R} \to \partial\Omega$. Choose $N_1 > 0$. Then there exists a deformation $(c_t)_{t\in\mathbf{R}_+}$ such that, for each $t \in \mathbf{R}_+$:*

1. *c_t is a convex curve in Ω, with curvature $\kappa \geq \kappa_m$, and $\kappa \geq \kappa_m + tc_1$ along $g_t([-N_1, N_1])$;*
2. *$g_t \equiv g_0$ outside $[-N_1 - M_1', N_1 + M_1']$;*
3. *for each $s \in \mathbf{R}$, either $(\partial_t g_t)(s)$ is zero, or its orthogonal is a support direction of g_t, and its norm is at most M_1'.*

Proof. The underlying idea is to apply corollary 7.6 recursively, to obtain the existence of such a deformation for $t \in [0, T]$ for some $T > 0$. The formal proof, however, has to be done in a slightly different way. Suppose that such a deformation can not exist for all $t \in \mathbf{R}_+$. Let E be the set of couples $(t, (g_s)_{s\in[0,t)})$, where $t > 0$ and $(g_s)_{s\in[0,t)}$ satisfies the conditions demanded, but only until time t.

There is a natural order on E, with:

$$(t, (g_s)_{s\in[0,t)}) \leq (t', (g_s')_{s\in[0,t')})$$

if $t \leq t'$ and $g_s = g_s'$ for $s \leq t$. E has a maximal element, say $(t_0, (g_s^0)_{s\in[0,t_0)})$. For $u \in \mathbf{R}$, let:

$$g_{t_0}^0(u) := \lim_{t\to t_0} g_t^0(u) \in \overline{\Sigma} .$$

Because of the convexity of the (g_s^0) and because $\partial_{I\!I\!I}\Sigma$ has no concave point, g_{t_0} is a convex curve. Thus one can apply corollary 7.6 to g_{t_0}, and this contradicts the maximality of $(t_0, (g_s^0)_{s\in[0,t_0)})$. □

We now consider related questions for deformations of curves which are topologically S^1, and which are not necessarily convex, but have curvature bounded from below.

Corollary 7.8. *There exist $c_2 > 0$ and $M_2' \geq 1$ as follows. Let $\kappa_m \in \mathbf{R}$, $L \in \mathbf{R}_+^*$, and let $g_0 : \mathbf{R}/L\mathbf{Z} \rightarrow \Sigma$ be a curve parametrized at speed one, with curvature $\kappa \geq \kappa_m$ (with the normal oriented towards the non-compact side of g_0). Then there exist $T_2 > 0$ and a deformation $(g_t)_{t\in[0,T_2]}$ such that, for each $t \in [0, T_2]$:*

1. *g_t is a curve which bounds a compact set, with curvature $\kappa \geq \kappa_m + tc_2$;*
2. *for each $s \in [0, L]$, $\|(\partial_t g_t)(s)\| \in [1, M_2']$.*

Proof. It is similar to the proof of corollary 7.6. First choose a sequence of smooth curves $g_n : \mathbf{R}/L\mathbf{Z}$ converging to g_0, such that g_n is in the interior of g_m for $n \leq m$ and that the curvature κ_n of g_n is at least $\kappa_m - \alpha_n$, with $\lim_{n\rightarrow\infty} \alpha_n = 0$.

For $n \in \mathbf{N}^*$ and $s \in \mathbf{R}/L\mathbf{Z}$, let:

$$u_n(s) := \tau(Jg_n'(s)) ,$$

where we suppose again that Jg_n' is towards the non-compact side of g_n. Let $\tilde{u}_n$ be the lift of u_n to a function on $\mathbf{R}$. Apply proposition 7.5 to $\tilde{u}_n$, to obtain a piecewise smooth, Lipschitz function $\tilde{y}_n : \mathbf{R} \rightarrow M_1'$ which vanishes outside $[-L - M_1', L + M_1']$ and which is at least 1 on $[-L, L]$. Let $y_n : \mathbf{R}/L\mathbf{Z} \rightarrow \mathbf{R}$ be the function defined by:

$$y_n(u) := \sum_{s\in u} \tilde{y}(s) ,$$

where only finitely many terms are non-zero by definition of $\tilde{y}_n$. After multiplying it by a constant, y_n is a Lipschitz, piecewise smooth function from $\mathbf{R}/L\mathbf{Z}$ to $[1, M_0']$ (for some $M_0' > 1$ which depends on L and on M_1') which is a solution of (5).

The rest of the proof can be done quite like in the proof of corollary 7.6, so we leave the details to the reader. □

Corollary 7.9. *Corollary 7.8 is true for any $T_2 > 0$.*

Proof. Like the proof of corollary 7.7 from corollary 7.6. □

We now have enough results on the deformations of curves, and we turn to another simple property: a convex, complete curve which separates a convex subset of Σ into two parts can not be "too" curved.

Proposition 7.10. *There exists a constant $\kappa_0(K_4, \tau_0)$ as follows. Let Ω be a closed, locally convex subset of Σ, and let $\rho : \mathbf{R} \rightarrow \Omega$ be a convex, injective curve, parametrized at speed one, which separates Ω into two connected components, and with curvature:*

$$\kappa = \kappa_1 + \kappa_m ,$$

where $\kappa_1 > 0$ is a constant and κ_m is a positive measure. Then $\kappa_1 \leq \kappa_0$.

Proof. First note that, by a direct approximation argument, it is enough to prove the result when ρ is smooth, so we suppose that is the case. Let $t_0 \in \mathbf{R}$. By corollary 3.4, there exists $\epsilon > 0$ (depending only on K_5 and τ_0), such that $\exp^{\tilde{\nabla}}_{\rho(t_0)}$ is a diffeomorphism from the subset of the ball of radius ϵ where it is defined onto its image. Therefore, for all $t \in [t_0, t_0 + \epsilon]$, there exists a unique $\tilde{\nabla}$-geodesic $\gamma_t : [0, L_t] \rightarrow \Omega$ of minimal length between $\rho(t_0)$ et $\rho(t)$. For $t \in [t_0, t_0 + \epsilon]$, let $\theta_1(t)$ be the angle between $\rho'(t_0)$ and $\gamma_t'(0)$, $\theta_2(t)$ the angle between $\gamma_t'(L_t)$ and $\rho'(t)$, $D(t)$ the domain in Ω bounded by $\rho([t_0, t])$ and by γ_t, and $A(t)$ its area.

From the Gauss-Bonnet theorem, for each $t \in [t_0, t_0 + \epsilon]$:

$$\begin{aligned} \theta_1 + \theta_2 &= \int_{t_0}^{t_1} \kappa(ds) + \int_{D(t)} \tilde{K} da \\ &\geq \kappa_1(t - t_0) + K_4 A(t) . \end{aligned}$$

But it is easy to check, using equation (5) of corollary 3.2, that, if L_t remains small enough (so that ϵ remains smaller than a constant depending only on K_5 and on τ_0), then $A(t)$ is bounded by:

$$A(t) \leq \int_{t_0}^{t} L(s)ds \leq 2 \int_{t_0}^{t} (s - t_0)ds \leq (t - t_0)^2 .$$

As a consequence of those two equations, if κ_1 is large enough (larger than a constant depending only on K_5 and on τ_0), there exists $t_1 \in [t_0, t_0 + \epsilon]$ such that:

$$\theta_1(t_1) + \theta_2(t_1) = 2\pi \ .$$

Then there exists $t_2 \in [t_0, t_1]$ verifying one of the following properties:

1. either $\theta_1(t_2) = \pi$;
2. or $\theta_1(t_2) \leq \pi$ and $\theta_2(t) = \pi$.

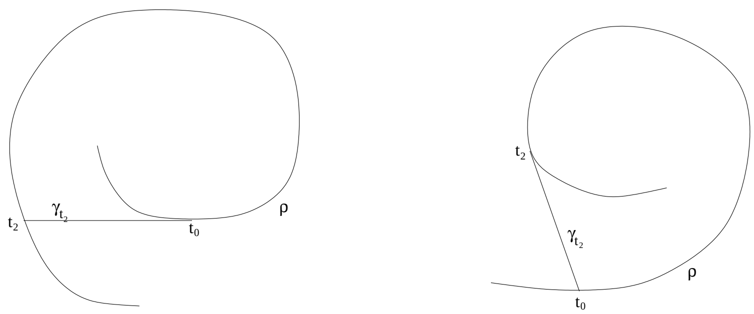

Figure 7.3

Recall that ρ is injective; therefore, using (5) and the upper bound (5) on $A(t_2)$, one sees that, if ϵ is small enough (smaller than a constant which this time depends on K_4 and on τ_0), then:

- in the first case, that $\rho(]-\infty, t_0])$ remains in the domain of Ω bounded by $\rho([t_0, t_2])$ and by γ_{t_2};
- in the second case, that $\rho([t_2, \infty])$ remains in the domain of Ω bounded by γ_{t_2}.

In both cases, "half" of ρ remain in a compact domain of Ω, and therefore ρ can not separate Ω in two parts. $\square$

We can show that Ω can not be complete:

Proof of lemma 7.2. Choose a smooth, simple closed curve γ_0 in Ω, with its unit normal oriented towards the non-compact connected component of $\Omega \setminus \gamma_0$ (there exists one because Ω is complete and simply connected). Then the geodesic curvature of γ_0 can be written as:

$$\kappa = -\kappa_1 + \kappa_m \ ,$$

with $\kappa_1 \in \mathbf{R}_+$ and $\kappa_m \geq 0$. Apply corollary 7.9, to obtain a continuous family $(\gamma_t)_{t\in\mathbf{R}_+}$ of curves with curvature bounded below by $-\kappa_1 + c_2 t$. For t large enough, this contradicts proposition 7.10. $\square$

Finally, we can now prove that $\partial\Omega$ can not contain any non-compact curve.

Proof of lemma 7.3. Suppose that $\partial_{I\!I\!I}\Omega$ contains a non compact curve g_0. Since Ω is simply connected, proposition 3.7 shows that g_0 is at non-zero distance from the other connected components of $\partial_{I\!I\!I}\Omega$.

Corollary 7.7 thus shows that there exists a continuous deformation $(g_t)_{t\in[0,T)}$ of g_0 in Ω, which goes on until time T with either $T = \infty$ or $\lim_{t\to T} d_{I\!I\!I}(g_t, \partial\Omega \setminus g_0) = 0$. But proposition 3.7 excludes this possibility, so that $T = \infty$. This contradicts proposition 7.10 just as in the proof of lemma 7.2 above. $\square$

It is a natural question whether the hypothesis we had to make concerning the relationship between τ_0 and K_4 are really necessary to obtain a bound on the area of $(\Omega, I\!I\!I)$. This is all the more important since the main geometric result of this paper, theorem 0.2, is limited precisely by this hypothesis.

The following example shows that this relation is in a sense optimal. Note that, for homogeneity reasons, any relationship between the curvature and the torsion should relate the curvature to the square of the torsion.

Consider the hyperbolic plane H^2, with the connection ∇_t obtained by adding to the Levi-Civita connection ∇_0 a 1-form β_t, so that:

$$\nabla^t_X Y = \nabla^0_X Y + \beta_t(X) JY \ ,$$

with $\beta_t = -tu^*_\theta$, where u_θ is at each point the unit normal bundle to the geodesic coming from 0 (with the usual orientation) and u^*_θ is the dual 1-form.

It is easy to check the following points:

- the torsion τ_t of ∇_t is such that: $\|\tau_t\| = t$;
- for $\beta > 1$ and r large enough, the complement Γ_r of the ball centered at 0 of radius r is ∇_t-convex, with infinite area;
- the curvature of ∇_t is: $K_t = t.\mathrm{th}(r) - 1$ at distance r from 0.

The lower bound on K_t is therefore $K^m_t = t-1$, and the smallest value of τ_t^2/K^m_t is obtained for $t = 2$, it is equal to $\frac{1}{4}$. So, using the same notations as above, for $K_4 = \tau_0^2/4$, there is already a counter-example to the results proved above for $K_4 > \tau_0^2/4$.

An interested reader might check that one can built other examples, based on deformations of the Levi-Civita connection of a positively curved surface (e.g. an annulus in S^2 with its canonical metric) such that convex domains with infinite area, for instance “strips” between two convex curves with constant curvature, exist. But the limiting relations between K_4 and τ_0 are the same as in the hyperbolic-based example above.

8 Some examples and further statements

This section contains some other, more precise results like theorem 0.2, and also some examples which indicate that theorem 0.2 is, in some sense, optimal.

First note that the proof which was given actually shows a little more than what was stated, namely:

Theorem 8.1. *Let $K_1, K_2, K_3 \in \mathbf{R}$ be such that $K_1 < 0$ and that $K_1 < K_2 \leq K_3$. Let (Σ, σ) be a complete Riemannian surface, and let (M, μ) be a Riemannian 3-manifold. Suppose that the curvature K_Σ of Σ is bounded above by K_1, and that, for all $m \in M$, the maximal and minimal curvatures of the 2-planes in T_mM, K_M and K_m, are in $[K_2, K_3]$ and such that:*

$$\frac{K_M - K_m}{2\sqrt{(K_1 - K_M)(K_1 - K_m)}} \leq \tau_0 \ ,$$

with

$$\tau_0^2 < \frac{4K_1}{K_1 - K_3} \ \textit{if}\ K_3 \leq 0 \ , \ \tau_0^2 < 4 \ \textit{if}\ K_3 \geq 0 \ .$$

Suppose further that the gradient of the sectional curvature of (M, μ) (on M) and the gradient of $(K_\sigma)^{-1/2}$ (on Σ) are bounded. Then there exists no C^3 isometric immersion of (Σ, σ) into (M, μ).

Note that the precise value of K_2 plays no role in this statement; but we need to know that $K_2 > K_1$ to obtain an upper bound on the curvature of $\tilde{\nabla}$. This hypothesis is related to the “uniform hyperbolicity” of the immersion.

Theorem 0.2 is strongly related to a well-knowsn family of PDEs, the Monge-Ampère PDEs of hyperbolic type. Those are usually written, on a domain $\Omega \subset \mathbf{R}^2$, as:

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x \partial y} = -b \ ,$$

where b might be a function on Ω, or depend on u and maybe of its first derivatives.

Equation (5) can be written as an equation on the bundle morphism H associated to the hessian $\nabla^2 u$ of u (by: $(\nabla^2 u)(X,Y) = \langle HX|Y\rangle = \langle X|HY\rangle$) with first order conditions meaning that H is the hessian of a function; we get thus:

$$\left\{ \begin{array}{c} \det(H) = -b \\ d^\nabla H = 0 \ , \end{array} \right.$$

This form seems well adapted to the study of Monge-Ampère equations over Riemannian surfaces other than $\mathbf{R}^2$. in particular, if b is everywhere equal to $K_\sigma - K_0$, then this equation is verified by the second fundamental forms of the immersions into space-forms with curvature K_0. For immersions into a 3-manifold with non-constant curvature, a right-hand side appears in the first equation of (5). This generalization is not equivalent to what is obtained by writing (5) in term of the hessian of u; if (M,μ) is a Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R, if $u : M \rightarrow \mathbf{R}$ is a C^2 function, and if H is the bundle morphism associated to its hessian, then:

$$\forall m \in M, \ \forall X, Y \in T_m M, \ (d^\nabla H)(X,Y) = -R(X,Y)(Du) \ ,$$

which differs from (5) because the 1-jet of u appears.

One can check that the proof given for theorem 0.2 also proves the following:

Theorem 8.2. *Let $\epsilon_0 > 0$, $0 < b_m \leq b_M$ and $\tau_0 \geq 0$ be such that $b_M \tau_0^2 < 4\epsilon_0 b_m^2$; let (Σ, σ) be a complete Riemannian surface with curvature $K \leq -\epsilon_0$, $b : \Sigma \rightarrow [b_m, b_M]$ be a C^1 function with bounded gradient on Σ, and let τ be a C^0 vector field on Σ with $\|\tau\| \leq \tau_0$. Then the system:*

$$\left\{ \begin{array}{c} \det(H) = -b \\ d^\nabla H = \tau \otimes \nu_\sigma \end{array} \right.$$

(where ν_σ is the area form associated to σ and H is a symmetric endomorphism field on Σ) has no C^1 solution on Σ.

The point is that such a solution H would allow the definition of a "virtual third fundamental form" on Σ as: $I\!I\!I(X,Y) = \sigma(HX, HY)$. In addition, we could define a connection $\tilde{\nabla}$ (as in section 2) compatible with $I\!I\!I$, with curvature $\tilde{K} = -K/b$, and torsion $\tilde{\tau} = -H^{-1}\tau/b$. With the hypotheses of theorem 8.2, we would have:

$$\|\tilde{\tau}\|_{I\!I\!I} = \|\tau\|_\sigma / b \leq \tau_0 / b_m \ ,$$

so that:

$$\|\tilde{\tau}\|^2_{I\!I\!I} \leq \frac{\tau_0^2}{b_m^2} < \frac{4\epsilon_0}{b_M} \leq 4\tilde{K} \ ,$$

and the analog of lemma 1.4 would indicate that $(\Sigma, I\!I\!I, \tilde{\nabla})$ should be convex; while the analog of lemma 1.9 would lead to a contradiction.

There are also various possible improvements of theorem 0.2. For instance, almost all the proof takes place "at infinity", while the interior of $(\Sigma, I\!I\!I, \tilde{\nabla})$ is important essentially only in the proof of lemma 1.9. Therefore, one can check that, if (Σ, σ) is simply connected but satisfies the hypothesis of theorem 0.2 only outside a compact set, then isometric immersions remain impossible.

It is natural to wonder to what extend the conditions in theorem 0.2 are really necessary. It is not clear concerning the hypothesis that the gradient of the sectional curvatures of (Σ, σ) and of (M, μ) are bounded, but the following example shows that the inequalities in theorem 0.2 are necessary.

Let g_λ be the symmetric 2-form defined on $\mathbf{R}^3$ as:

$$g_\lambda = (1 + 2\lambda z)\cosh^2(y)\cosh^2(z) dx \otimes dx + (1 - 2\lambda z)\cosh^2(z) dy \otimes dy + dz \otimes dz \ ,$$

for $\lambda > 0$. It is a Riemannian metric in a neighborhood of $P_0 = \{(x,y,z) \in \mathbf{R}^3 \mid z = 0\}$. Let V_ϵ be such a neighborhood:

$$V_\epsilon = \{(x,y,z) \in \mathbf{R}^3 \mid |z| \leq \epsilon\} \ .$$

When $\lambda = 0$, g_λ is hyperbolic (i.e. it has constant curvature -1.

$g_{\lambda|P_0}$ is hyperbolic, so we have an isometric embedding of H^2 into a Riemannian manifold with boundary $(V_\epsilon, g_{\lambda|V_\epsilon})$.

A rather boring computation leads to the following expression of the Riemann curvature tensor of g_λ: if (e_1, e_2, e_3) is a orthonormal moving frame made of vectors directed by $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$, then

$$\begin{aligned} g_\lambda(R_\lambda(e_1,e_2)e_1,e_2) &= \lambda^2 - 1 \\ g_\lambda(R_\lambda(e_1,e_3)e_1,e_3) &= \lambda^2 - 1 \\ g_\lambda(R_\lambda(e_3,e_2)e_3,e_2) &= \lambda^2 - 1 \\ g_\lambda(R_\lambda(e_1,e_2)e_1,e_3) &= 2\lambda\tanh(y) \\ g_\lambda(R_\lambda(e_2,e_1)e_2,e_3) &= 0 \\ g_\lambda(R_\lambda(e_3,e_1)e_3,e_2) &= 0\ . \end{aligned}$$

It is then a matter of computation to check that, at the point (x, y, z), the eigenvalues of R_λ remain in the interval $[\lambda^2 - 1 - 2\lambda\tanh(y), \lambda^2 - 1 + 2\lambda\tanh(y)]$. Therefore, for the isometric embedding of H^2 into $(V_\epsilon, g_{\lambda|V_\epsilon})$ which we have obtained:

$$K_1 = -1,\ K_2 = \lambda^2 - 1 - 2\lambda,\ K_3 = \lambda^2 - 1 + 2\lambda\ ,$$

so that, for λ large enough:

$$(K_3 - K_2)^2 = 16\lambda^2$$

and

$$16(K_2 - K_1)|K_1| = 16\lambda^2 - 32\lambda\ .$$

As $\lambda \to \infty$, we get very chose to the inequalities in theorem 0.2. This example is actually related to the one built in section 7.

The condition that $K_1 < 0$ is also necessary, because of a classical example: $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ admits an isometric embedding into S^3 with its canonical metric (up to a factor $\sqrt{2}$. This is because T^2 can be obtained as:

$$T^2 = \{(x, y, z, t) \in \mathbf{R}^4 \mid x^2 + y^2 = z^2 + t^2 = \frac{1}{2}\}$$

with the induced metric, while S^3 is:

$$S^3 = \{(x, y, z, t) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$$

and the embedding of T^2 in S^3 follows.

If the target manifold is Lorentzian instead of Riemannian, the proof given in this paper also applies, with the necessary changes in the hypothesis concerning the curvature of Σ and of M. In fact, the hypothesis which are needed in this case are such that the only possible target manifolds are those with constant negative curvature. This leads to the following result, which was already given in [Sch99]:

Theorem 8.3. *Let $\epsilon \in]0, -1/2[$, and let (Σ, σ) be a complete Riemannian surface with curvature K between $-1 + \epsilon$ and $-\epsilon$, and such that the gradient K is bounded. Then there exists no C^3 isometric immersion of (Σ, σ) into the anti-de Sitter space H^3_1.*

On the other hand, the following assertion is easy:

Proposition 8.4. *Let (Σ, σ) be a complete Riemannian surface whose curvature K is larger than some $\epsilon > 0$. There is no isometric immersion ϕ of (Σ, σ) into a Lorentzian manifold (M, μ) such that, at each $s \in \Sigma$:*

$$K^\Sigma(s) > K^M(\phi_*(T_s\Sigma))\ .$$

As a consequence, there is no strictly hyperbolic isometric immersion of a complete surface into the Minkowski or the de Sitter space.

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